Resurgence and Phase Transitions

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KEK Theory Workshop, December 4, 2019

GD & Mithat Ünsal, review: 1603.04924
A. Ahmed & GD: arXiv:1710.01812
GD, arXiv:1901.02076

O.Costin & GD, 1904.11593, ...

[DOE Division of High Energy Physics]

Physical Motivation: Resurgence and Quantum Field Theory

- non-perturbative definition of QFT
- Minkowski vs. Euclidean QFT
- "sign problem" in finite density QFT
- dynamical & non-equilibrium physics in path integrals
- phase transitions (Lee-Yang and Fisher zeroes)
- common thread: analytic continuation of path integrals

- question: does resurgence give (useful) new insight?

The Big Question

• Can we make physical, mathematical and computational sense of a Lefschetz thimble expansion of a path integral?

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right)$$

$$\text{"} = \text{"} \sum_{\text{thimble}} \mathcal{N}_{\text{th}} \, e^{i\,\phi_{\text{th}}} \int_{\text{th}} \mathcal{D}A \times (\mathcal{J}_{\text{th}}) \times \exp\left(\mathcal{R}e\left[\frac{i}{\hbar} S[A]\right]\right)$$

• $Z(\hbar) \to Z(\hbar, \text{masses}, \text{couplings}, \mu, T, B, \ldots)$

• $Z(\hbar) \to Z(\hbar, N)$, and $N \to \infty$ for a phase transition

• since we <u>need</u> complex analysis and contour deformation to make sense of oscillatory ordinary integrals, it is natural to explore similar methods for path integrals

• resurgence and Stokes transitions: transmutation of trans-series structures across phase transitions

- the physics message from Écalle's resurgence theory: different critical points are related in subtle and powerful ways

Borel summation: extracting physics from asymptotic series

Borel transform of series, where $c_n \sim n!$, $n \to \infty$

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \quad \longrightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has a finite radius of convergence

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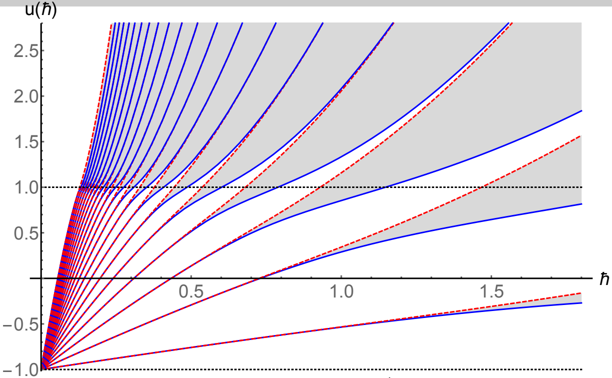
Borel summation of original asymptotic series:

$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

- the singularities of $\mathcal{B}[f](t)$ provide a physical encoding of the global asymptotic behavior of $f(g)$, which is also much more mathematically efficient than the asymptotic series
- Borel singularities $\leftrightarrow$ non-perturbative physical objects
- resurgence: isolated poles, algebraic & logarithmic cuts

Mathieu Equation Spectrum: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$



$$u_\pm(\hbar,N)=u_{\rm pert}(\hbar,N)\pm\frac{\hbar}{\sqrt{2\pi}}\frac{1}{N!}\left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}}\exp\left[-\frac{8}{\hbar}\right]\mathcal{P}_{\rm inst}(\hbar,N)+\dots$$

$$\mathcal{P}_{\rm inst}(\hbar,N)=\frac{\partial u_{\rm pert}(\hbar,N)}{\partial N}\exp\left[S\int_0^\hbar\frac{d\hbar}{\hbar^3}\left(\frac{\partial u_{\rm pert}(\hbar,N)}{\partial N}-\hbar+\frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right]$$

all non-perturbative effects encoded in perturbative expansion

GD & Ünsal (2013); Başar, GD & Ünsal (2017): applies to bands & gaps

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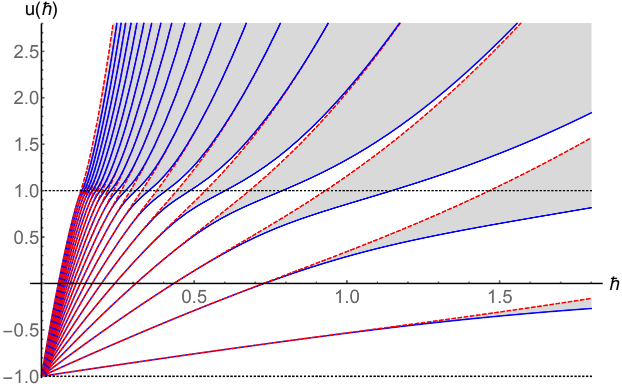
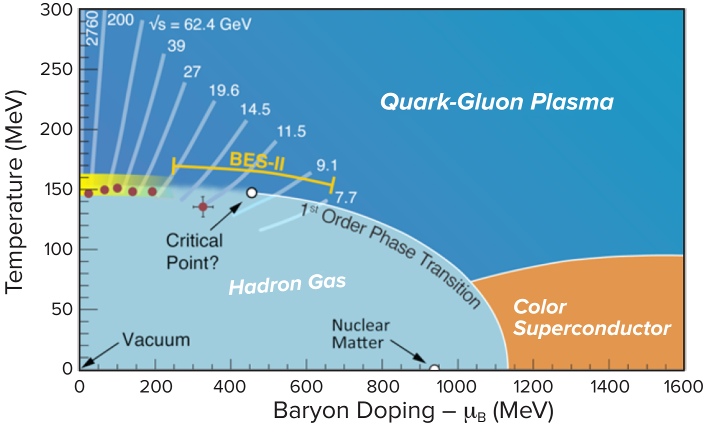


- phase transition at $\hbar N=\frac{8}{\pi}$: narrow bands vs. narrow gaps
- real vs. complex instantons (Dykhne, 1961; Başar/GD)
- phase transition = "instanton condensation"
- maps to $\mathcal{N}=2$ SUSY QFT (Nekrasov et al, Mironov et al; Başar/GD)

Physical Motivation: QCD phase diagram



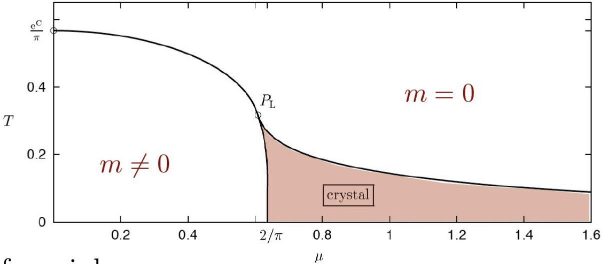
- sign problem: "complex probability" at finite baryon density?

$$\int \mathcal{D}A\, e^{-S_{YM}[A]+\ln\det(\not{D}+m+i\,\mu\gamma^0)}$$

Phase Transition in 1+1 dim. Gross-Neveu Model

$$\mathcal{L} = \bar{\psi}_a i\partial\!\!\!/ \psi_a + \frac{g^2}{2}\left(\bar{\psi}_a \psi_a\right)^2$$

- asymptotically free; dynamical mass; chiral symmetry
- large N_f chiral symmetry breaking phase transition
- physics = (relativistic) Peierls instability in 1 dimension



- saddles from inhomogeneous gap eqn. (Basar, GD, Thies, 2011)

$$\sigma(x;T,\mu) = \frac{\delta}{\delta\sigma(x;T,\mu)} \ln\det\left(i\,\partial\!\!\!/ - \sigma(x;T,\mu)\right)$$

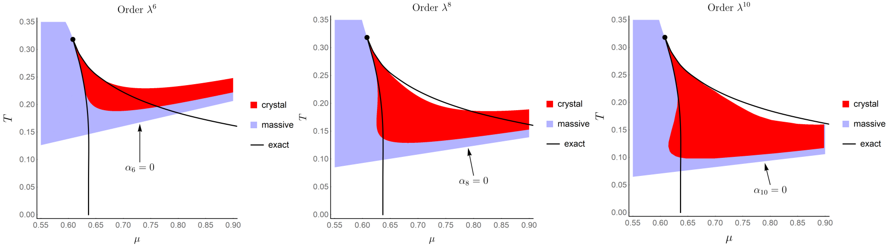
Phase Transition in 1+1 dim. Gross-Neveu Model

- thermodynamic potential

$$\Psi[\sigma; T, \mu] = -T \int dE \, \rho(E) \, \ln\left(1 + e^{-(E-\mu)/T}\right)$$
$$= \sum_n \alpha_n(T, \mu) f_n[\sigma(x; T, \mu)]$$

- (divergent) Ginzburg-Landau expansion = mKdV
- saddles: $\sigma(x) = \lambda \, \mathrm{sn}(\lambda \, x; \nu)$
- successive orders of GL expansion reveal the full crystal phase

(Basar, GD, Thies, 2011; Ahmed, 2018)



Phase Transition in 1+1 dim. Gross-Neveu Model

- most difficult point: $\mu_c = \frac{2}{\pi}$, $T = 0$
- high density expansion at $T = 0$: (convergent !)

$$\mathcal{E}(\rho) \sim \frac{\pi}{2}\,\rho^2 \left(1 - \frac{1}{32(\pi\rho)^4} + \frac{3}{8192(\pi\rho)^8} - \dots \right)$$

- low density expansion at $T = 0$: (non-perturbative !)

$$\mathcal{E}(\rho) \sim -\frac{1}{4\pi} + \frac{2\rho}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{e^{-k/\rho}}{\rho^{k-2}} \mathcal{F}_{k-1}(\rho)$$

(Thies; 2004; GD, 2018)

- resurgent trans-series
- analogous expansions at fixed T/μ

Phase Transition in 2d Lattice Ising Model

- diagonal correlation function: $C(s, N) = \langle \sigma_{0,0} \, \sigma_{N,N} \rangle (s)$
- $C(s, N)$ = tau function for Painlevé VI (Jimbo, Miwa, 1980)
- simple Toeplitz det representation ("linearizes")
- scaling limit: $N \to \infty$ & $T \to T_c$: PVI $\to$ PIII (McCoy et al)

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- convergent and resurgent (!) conformal block expansions at high and low T (Jimbo; Lisovyy et al; Bonelli et al; GD)

$$\tau(t) \sim \sum_{n=-\infty}^{\infty} s^n \, C(\vec{\theta}, \sigma + n) \, \mathcal{B}(\vec{\theta}, \sigma + n; t)$$

$$\mathcal{B}(\vec{\theta}, \sigma; t) \propto t^{\sigma^2} \sum_{\lambda, \mu \in \mathcal{Y}} \mathcal{B}_{\lambda,\mu}(\vec{\theta}, \sigma) t^{|\lambda| + |\mu|}$$

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- resurgence applies also to convergent expansions (!)

Other Examples: Phase Transitions

- particle-on-circle: sum over spectrum versus sum over winding (saddles) (Schulman, 1968)
- Bose gas (Cristoforetti et al, Alexandru et al)
- Thirring model (Alexandru et al)
- Hubbard model (Tanizaki et al; ...)
- Hydrodynamics: short/late-time (Heller et al; Aniceto et al; Basar/GD)
- Large N matrix models (Mariño, Schiappa, Couso, Putrov, Russo, ...)
- Painlevé (Jimbo et al; Its et al; Lisovyy et al; Litvinov et al; Costin, GD)
- Gross-Witten-Wadia model (Mariño; Ahmed, GD)
- resurgence and superconductors (Mariño, Reis)
- ...

$$Z(g^2, N) = \int_{U(N)} DU \, \exp\left[\frac{1}{g^2} \mathrm{tr}\left(U + U^\dagger\right)\right]$$

- one-plaquette matrix model for 2d lattice Yang-Mills
- two variables: g^2 and N ('t Hooft coupling: $t \equiv g^2 N/2$)
- 3rd order phase transition at $N = \infty$, $t = 1$ (universal!)
- double-scaling limit: Painlevé II
- physics of phase transition = condensation of instantons
- random matrix theory/orthogonal polynomials result:

$$Z(g^2, N) = \det\left(I_{j-k}(x)\right)_{j,k=1,\dots N} \qquad , \quad x \equiv \frac{2}{g^2}$$

Gross-Witten-Wadia $N = \infty$ Phase Transition

3rd order transition: kink in the specific heat

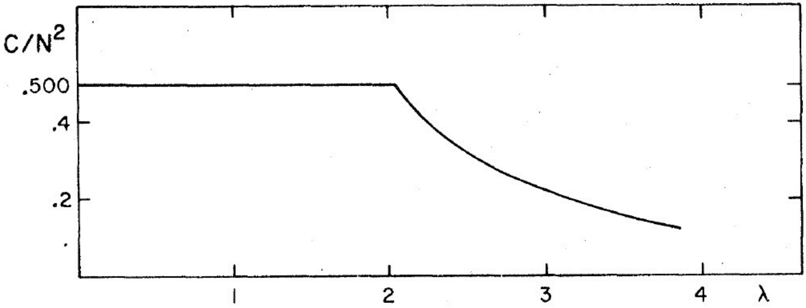

FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

D. Gross, E. Witten, 1980

- what about non-perturbative large N effects?

Resurgence in Gross-Witten-Wadia Model: Transmutation of the Trans-series

Ahmed & GD: 1710.01812

- "order parameter": with 't Hooft coupling $t \equiv \frac{1}{2} N g^2$

$$\Delta(t, N) \equiv \langle \det U \rangle = \frac{\det \left[I_{j-k+1}\left(\frac{N}{t}\right)\right]_{j,k=1,\dots,N}}{\det \left[I_{j-k}\left(\frac{N}{t}\right)\right]_{j,k=1,\dots,N}}$$

- for any N, $\Delta(t, N)$ satisfies a Painlevé III equation:

$$t^2 \Delta'' + t\Delta' + \frac{N^2 \Delta}{t^2}\left(1-\Delta^2\right) = \frac{\Delta}{1-\Delta^2}\left(N^2 - t^2\left(\Delta'\right)^2\right)$$

- weak-coupling expansion is a divergent series:
  → trans-series non-perturbative completion

- strong-coupling expansion is a convergent series:
  but it still has a non-perturbative completion !

- N is now a parameter, not necessarily integer !

Resurgence: Large N 't Hooft limit at Weak Coupling

- large N trans-series at weak-coupling ($t \equiv N/x < 1$)

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{i}{2\sqrt{2\pi N}}\,\sigma_{\text{weak}}\frac{t\, e^{-N S_{\text{weak}}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \dots$$

- large N weak-coupling action

$$S_{\text{weak}}(t) = \frac{2\sqrt{1-t}}{t} - 2\,\text{arctanh}\left(\sqrt{1-t}\right)$$

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- large-order growth of perturbative coefficients ($\forall\, t < 1$):

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\,\frac{\Gamma(2n-\frac{5}{2})}{(S_{\rm weak}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\,\frac{S_{\rm weak}(t)}{(2n-\frac{7}{2})} + \dots\right.$$

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- (parametric) resurgence relations, for all t:

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{1}{N} + \dots$$

Resurgence: Large N 't Hooft limit at Strong Coupling

- large N transseries at strong-coupling: $\Delta(t,N) \approx \sigma J_N\left(\frac{N}{t}\right)$

$$\Delta(t,N) = \sum_{k=1,3,5,\ldots}^{\infty} (\sigma_{\text{strong}})^k \Delta_{(k)}(t,N)$$

  - "Debye expansion" for Bessel function: $J_N(N/t)$

$$\Delta(t,N) \sim \frac{\sqrt{t}\, e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n(t)}{N^n} + \frac{1}{4(t^2-1)} \left(\frac{\sqrt{t} e^{-N S_{\text{strong}}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}(t)}{N^n} + \ldots$$

- large N strong-coupling action: $S_{\text{st}}(t) = \text{arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$

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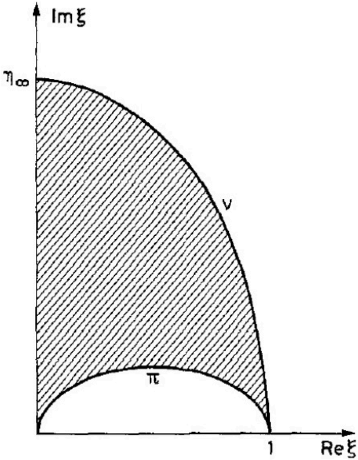
- large N strong-coupling action: $S_{\text{st}}(t) = \text{arccosh(t)} - \sqrt{1-\frac{1}{t^2}}$

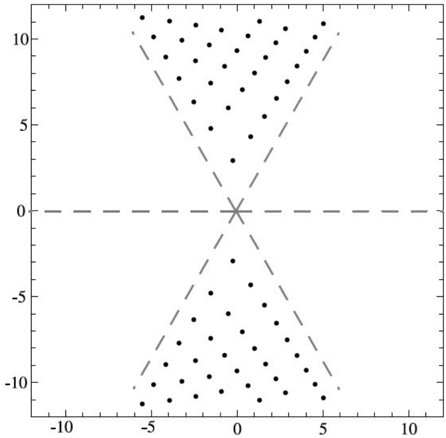
  - large-order/low-order (parametric) resurgence relations:

$$U_n(t) \sim \frac{(-1)^n (n-1)!}{2\pi (2S_{\text{strong}}(t))^n} \left(1 + U_1(t) \frac{(2S_{\text{strong}}(t))}{(n-1)} + U_2(t) \frac{(2S_{\text{strong}}(t))^2}{(n-1)(n-2)} + \right.$$

Gross-Witten-Wadia Phase Transition and Lee-Yang zeros

Lee-Yang: complex zeros of Z pinch the real axis at the phase transition point in the thermodynamic limit





GWW zeros (Kolbig)

Painlevé II (Novokshenov; Huang)

• resurgence suggests that local analysis of perturbation theory encodes global information

• Questions:
How much global information can be decoded from a FINITE number of perturbative coefficients ?
How much information is needed to see and to probe phase transitions ?

• resurgent functions have orderly structure in Borel plane
$\Rightarrow$ develop extrapolation and summation methods that take advantage of this!

• high precision test for Painlevé I (but integrability is not important for the method)

• general & explicit large N estimates (Costin, GD; to appear)

Perturbative Expansion of Painlevé I Equation

- Painlevé I equation (double-scaling limit of 2d quantum gravity)

$$y''(x) = 6\, y^2(x) - x$$

- large x expansion:

$$y(x) \sim -\sqrt{\frac{x}{6}} \left(1 + \sum_{n=1}^{\infty} a_n \left(\frac{30}{(24\, x)^{5/4}} \right)^{2n} \right) \qquad , \qquad x \to +\infty$$

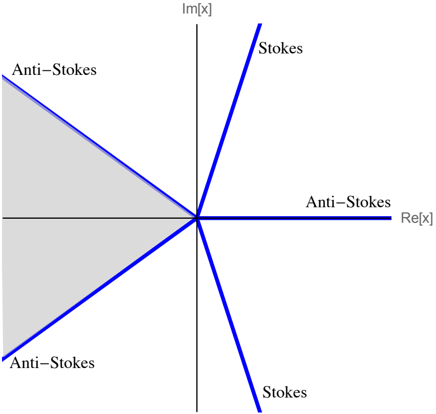
- perturbative input data: $\{a_1, a_2, \ldots, a_N\}$

$$\{\frac{4}{25}, -\frac{392}{625}, \frac{6\,272}{625}, -\frac{141\,196\,832}{390\,625}, \frac{9\,039\,055\,872}{390\,625}, \ldots, a_N\}$$

- this expansion defines the *tritronquée* solution to PI

Reconstruct global behavior from limited $x \to +\infty$ data?

- Painlevé I equation has inherent five-fold symmetry



- do our input coefficients (from $x = +\infty$) “know” this ?
- most interesting/difficult directions: phase transitions

High Precision at the Origin

O.Costin & GD, 1904.11593

- resurgence & Padé-Conformal-Borel transform
- "weak coupling to strong coupling" extrapolation
- $N = 50$ terms and Padé-Conformal-Borel input:

$y(0) \approx -0.187554308340494893838681757595832993231160909762138996933372651 67\ldots$

$y'(0) \approx -0.3049055602612288565341041249884896764031999134211283365005934 4290\ldots$

$y''(0) \approx 0.21105971146248859499298968451861337073253247206264082468899143841\ldots$

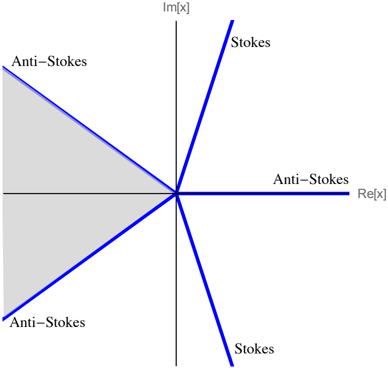
$$\left[y''(x) - 6y^2(x) + x\right]_{x=0} = O(10^{-65})$$

- best numerical integration algorithms $\rightarrow \quad \approx O(10^{-15})$
- WHY?
- Resurgent extrapolation method encodes global information about the function throughout the entire complex plane, not just along the positive real axis

Nonlinear Stokes Transition: the Tritronquée Pole Region

- Boutroux (1913): asymptotically, general Painlevé I solution has poles with 5-fold symmetry

- Dubrovin conjecture (2009): this asymptotic solution to Painlevé I only has poles in a $\frac{2\pi}{5}$ wedge



- proof: Costin-Huang-Tanveer (2012)

Stokes Transition: Mapping the Tritronquée Pole Region

- non-linear Stokes transitions crossing $\arg(x) = \pm\frac{4\pi}{5}$

O.Costin & GD, 1904.11593

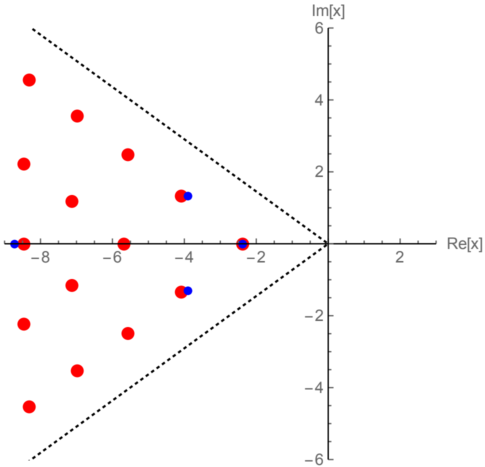


Figure: Complex poles: $N = 10$ (blue); $N = 50$ (red).

Metamorphosis: Asymptotic Series to Meromorphic Function

$$y(x) \approx \frac{1}{(x - x_{\rm pole})^2} + \frac{x_{\rm pole}}{10}(x - x_{\rm pole})^2 + \frac{1}{6}(x - x_{\rm pole})^3 + h_{\rm pole}(x - x_{\rm pole})^4 + \frac{x_{\rm pole}^2}{300}(x - x_{\rm pole})^6 + \dots$$

- our extrapolation ($y_N(x)$ with $N = 50$) near 1st pole:

$$\begin{aligned} y(x) \approx\ & \frac{0.9999999999999999999999999999999999997886}{(x - x_1)^2} \\ & + 3.5 \times 10^{-35} - 2.4 \times 10^{-34}(x - x_1) \\ & - 0.238416876956881663929914585244923803(x - x_1)^2 \\ & + 0.166666666666666666666666666666657864(x - x_1)^3 \\ & - 0.06213573922617764089649014164005140(x - x_1)^4 \\ & + 4 \times 10^{-31}(x - x_1)^5 \\ & + 0.0189475357392909503157755851627665(x - x_1)^6 + \dots \end{aligned}$$

- estimate approx 30 digit precision for x_1 and h_1

Conclusions

• Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series, which 'encode' analytic continuation information

• phase transitions $\leftrightarrow$ Stokes phenomenon

• QM, matrix models, differential/integral eqns

• numerical Lefschetz thimbles

• non-perturbative effects exist even for convergent series (e.g. periodic potential; Ising model; unitary matrix model; ...)

• resurgent extrapolation: non-perturbative information can be decoded from surprisingly little perturbative data

Applicable resurgent asymptotics: towards a universal theory

Participation in INI programmes is by invitation only. Anyone wishing to apply to participate in the associated workshop(s) should use the relevant workshop application form.

Programme
4th January 2021 to 25th June 2021