IR renormalon in a compactified spacetime: the case of the QCD(adj.) on $\mathbb{R}^3 \times S^1$

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- M. Ashie, O. Morikawa, H. Suzuki, H. Takaura and K. Takeuchi, arXiv:1909.05489 [hep-th], to appear in PTEP.
- K. Ishikawa, O. Morikawa, K. Shibata, H. Suzuki and H. Takaura, arXiv:1909.09579 [hep-th], to appear in PTEP.
- K. Ishikawa, O. Morikawa, A. Nakayama, K. Shibata, H. Suzuki and H. Takaura, arXiv:1908.00373 [hep-th].

Borel resummation of the perturbation series

- The coefficient in perturbative expansion,

$$f(\lambda) \sim \lambda \sum_{k=0}^{\infty} f_k \left(\frac{\beta_0 \lambda}{16\pi^2}\right)^k,$$

typically grows factorially as $k \to \infty$ (Dyson, Bender–Wu, Lipatov, ...),

$$f_k \sim b^{-k} k!.$$

- Perturbation series diverges and is an asymptotic series at best.
- Nevertheless, if the Borel transform,

$$B(u) := \sum_{k=0}^{\infty} \frac{f_k}{k!} u^k,$$

does not possess singularities on the positive real axis ($b < 0$),

$$f(\lambda) := \frac{16\pi^2}{\beta_0} \int_0^{\infty} du\, B(u)\, e^{-16\pi^2 u/(\beta_0 \lambda)},$$

defines the Borel sum.

Semi-classical understanding of the Borel singularity

- Assuming that $f(\lambda)$ is given by the functional integral as

$$f(\lambda) = \int \mathcal{D}\varphi \, e^{-\tilde{S}[\varphi]/\lambda},$$

the Borel transform is given by

$$B(u) = \frac{\beta_0}{16\pi^2} \int \mathcal{D}\varphi \, \delta(u - \beta_0 \tilde{S}[\varphi]/16\pi^2) \sim \sum_{\varphi_i} \left(\frac{\delta \tilde{S}[\varphi]}{\delta\varphi} \right)^{-1}_{\tilde{S}[\varphi_i]=16\pi^2 u/\beta_0}.$$

Thus the Borel transform develops a singularity at

$$u = \frac{\beta_0}{16\pi^2} \tilde{S}[\varphi_i], \qquad \varphi_i\text{: a solution of EoM.}$$

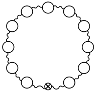
- A pair of instanton/anti-instanton, whose action in the $SU(N)$ gauge theory is

$$\tilde{S}[I\bar{I}] \sim 16\pi^2 N,$$

gives rise to Borel singularities at integer multiples of $u = \beta_0 N$.
- $k!$ associated with $I\bar{I}$ is attributed to the proliferation of the number of Feynman diagrams. This is suppressed in $N \to \infty$.

(IR) renormalon: another source of the factorial growth

- 't Hooft (1979): a single diagram that grows $\sim k!$
- This emerges from a diagram such as



and, in $\mathbb{R}^4$, evaluated as (we assume $\alpha > -2$)

$$
\begin{aligned}
&\sim \lambda \int \frac{d^4p}{(2\pi)^4}(p^2)^\alpha(-\ln p^2)^k\left(\frac{\beta_0\lambda}{16\pi^2}\right)^k \qquad \left(\beta_0 = \frac{11}{3} - \frac{2}{3}n_W\right)\\
&= \lambda\frac{1}{16\pi^2}(2+\alpha)^{-k-1}\int_{-\infty}^{\infty} dt\, e^{-t}t^k\left(\frac{\beta_0\lambda}{16\pi^2}\right)^k \qquad (t = -\ln p^2)\\
&\overset{t\sim k}{\sim} \lambda\frac{1}{16\pi^2}(2+\alpha)^{-k-1}k!\left(\frac{\beta_0\lambda}{16\pi^2}\right)^k.
\end{aligned}
$$

- This produces the Borel singularity at $u = 2 + \alpha$.
- In what follows, we consider a quantity with $\alpha = 0$.

Semi-classical understanding of the renormalon?

- In $\mathbb{R}^4$, the IR renormalon produces a Borel singularity at $u = 2$. The corresponding ambiguity in the Borel sum is

$$\left[e^{-16\pi^2/(\beta_0\lambda)}\right]^2 \sim \Lambda^4,$$

- Corresponding semi-classical object???

$$\tilde{S}[\varphi_i] \sim \frac{16\pi^2}{\beta_0} = \frac{1}{\beta_0 N}\tilde{S}[I\bar{I}]$$

- Argyres–Ünsal (arXiv:1206.1860) and Dunne–Ünsal (arXiv:1210.2423) argued that the so-called bion is the corresponding object.
- Still remains a conjecture for 4D QCD(adj.), for which

$$\tilde{S}[\text{bion}] \sim 16\pi^2 = \frac{1}{N}\tilde{S}[I\bar{I}], \qquad \beta_0 = \frac{11}{3} - \frac{2}{3}n_W.$$

- One may further push this picture in 2D $\mathbb{C}P^{N-1}$ model, for which

$$\beta_0 = 1.$$

IR renormalon and bion in compactified spaces

- The bion (pair of fractional instanton/anti-instanton) can exist only in compactified spaces with twisted boundary conditions (TBC).
- It is therefore important to study the IR renormalon in compactified spaces.
- Anber–Sulejmanpasic (arXiv:1410.0121), 4D $SU(2)$ and $SU(3)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$: vacuum polarization of the “photon” (see below) loses the $\ln p^2$ behavior. No IR renormalon!.
- Fujimori–Kamata–Misumi–Nitta–Sakai (arXiv:1810.03768), 2D SUSY $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$: very explicit bion calculus of the vacuum energy and observed the ambiguity corresponding to $u = 2$.
- Fujimori–Kamata–Misumi–Nitta–Sakai (arXiv:1607.04205), 1D $\mathbb{C}P^{N-1}$ SUSY QM: observed the coincidence between the bion calculus and the large order behavior of perturbation theory for the vacuum energy.
- Yamazaki–Yonekura (arXiv:1911.06327), $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with periodic boundary conditions: for finite R, convergent series of the coupling and no renormalon.
- No unified picture yet?

IR renormalon in compactified spaces

- Large N might give a clue. . .
- Ishikawa–Morikawa–Nakayama–Shibata–H.S.–Takaura (arXiv:1908.00373), 2D SUSY $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$, IR renormalon, but at $u = 3/2$, not $u = 2$! $\Rightarrow$ Morikawa's poster
- Ishikawa–Morikawa–Shibata–H.S.–Takaura (arXiv:1909.09579), This shift of the Borel singularity $u = 2 \to 3/2$ under the S^1 compactification is a very general phenomenon $\Rightarrow$ Takaura's talk
- Ashie–Morikawa–H.S.–Takaura–Takeuchi (arXiv:1909.05489), 4D $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$ with the large β_0 approximation (see below): IR renormalon, but at $u = 2$, not $u = 3/2$!
- Unfortunately, we do not have yet a general picture. . .

$SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$ with TBC

- $\mathbb{R}^3 \times S^1$:

$$(x_0, x_1, x_2) \in \mathbb{R}^3, \qquad 0 \leq x_3 < 2\pi R.$$

- Action ($\lambda_0 = g_0^2 N$: bare 't Hooft coupling)

$$S = -\frac{N}{2\lambda_0} \int d^4x \, \mathrm{tr}\left(\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}\right) - 2 \int d^4x \, \mathrm{tr}\left[\tilde{\bar{\psi}}(x)\gamma_\mu\left(\partial_\mu\tilde{\psi} + [\tilde{A}_\mu, \tilde{\psi}]\right)\right].$$

- $\mathbb{Z}_N$ twisted boundary conditions (TBC):

$$\tilde{\psi}(x_0, x_1, x_2, x_3 + 2\pi R) = \Omega\tilde{\psi}(x_0, x_1, x_2, x_3)\Omega^{-1},$$
$$\tilde{\bar{\psi}}(x_0, x_1, x_2, x_3 + 2\pi R) = \Omega\tilde{\bar{\psi}}(x_0, x_1, x_2, x_3)\Omega^{-1},$$
$$\tilde{A}_\mu(x_0, x_1, x_2, x_3 + 2\pi R) = \Omega\tilde{A}_\mu(x_0, x_1, x_2, x_3)\Omega^{-1},$$

where, denoting the Cartan generators by H_m,

$$\Omega = e^{i\frac{2\pi}{N}\phi\cdot H} = e^{i\pi\frac{N+1}{N}} \, \mathrm{diag}\left(e^{-i\frac{2\pi}{N}}\right)^j,$$

so that $\mathrm{tr}(e^{i\frac{2\pi}{N}}\Omega) = \mathrm{tr}\,\Omega$.

$SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$ with TBC

- Field variables in the Cartan–Weyl basis:

$$\tilde{\psi}(x) = -i \sum_{\ell=1}^{N-1} \tilde{\psi}^{\ell}(x) H_{\ell} - i \sum_{m \neq n} \tilde{\psi}^{mn}(x) E_{mn},$$

$$\tilde{\bar{\psi}}(x) = -i \sum_{\ell=1}^{N-1} \tilde{\bar{\psi}}^{\ell}(x) H_{\ell} - i \sum_{m \neq n} \tilde{\bar{\psi}}^{mn}(x) E_{mn},$$

$$\tilde{A}_{\mu}(x) = -i \sum_{\ell=1}^{N-1} \tilde{A}_{\mu}^{\ell}(x) H_{\ell} - i \sum_{m \neq n} \tilde{A}_{\mu}^{mn}(x) E_{mn}.$$

- We refer the Cartan part $\tilde{A}_{\mu}^{\ell}(x)$ to as the "photon", whereas the root part $\tilde{A}_{\mu}^{mn}(x)$ the "W-boson".

Gauge field propagators in the large β_0-approximation

- First, we extract a gauge-invariant set of diagrams, by considering the large flavor limit $n_W \to \infty$ (with $\lambda_0 n_W$ kept fixed).
- In this limit, the vacuum polarization is dominated by the fermion one-loop diagram and we have

$$\left\langle \tilde{A}^{\ell}_{\mu}(x)\tilde{A}^{r}_{\nu}(y)\right\rangle = \frac{\lambda}{N}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3\in\mathbb{Z}/R}$$

$$\times\, e^{ip(x-y)}\frac{1}{(p^2)^2}\left\{\left[(1-L)^{-1}\right]^{\ell r}p^2\mathcal{P}^{L}_{\mu\nu}+\left[(1-T)^{-1}\right]^{\ell r}p^2\mathcal{P}^{T}_{\mu\nu}+\delta^{\ell r}\frac{1}{\xi}p_\mu p_\nu\right\},$$

$$\left\langle \tilde{A}^{mn}_{\mu}(x)\tilde{A}^{pq}_{\nu}(y)\right\rangle = \frac{\lambda}{N}\delta^{mq}\delta^{np}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3\in\mathbb{Z}/R}$$

$$\times\left\{e^{ip(x-y)}\frac{1}{(p^2)^2}\left[(1-L)^{-1}p^2\mathcal{P}^{L}_{\mu\nu}+(1-T)^{-1}p^2\mathcal{P}^{T}_{\mu\nu}+\frac{1}{\xi}p_\mu p_\nu\right]\right\}_{p\to p_{mn}},$$

where $\mathcal{P}^{L,T}_{\mu\nu}$ are projection operators.

- It is important that the momentum for the W-boson is twisted as

$$p_{mn,\mu}\equiv p_\mu-\delta_{\mu 3}\frac{m-n}{RN}.$$

Gauge field propagators in the large β_0-approximation

- …, where λ and ξ are renormalized in the $\overline{\text{MS}}$ scheme,

$$L^{\ell r} \equiv \frac{\beta_0 \lambda}{16\pi^2}\left\{\delta^{\ell r}\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) + 12\sum_{j\neq 0}(\sigma_{j,N})_{\ell r}\int_0^1 dx\, e^{ixp_3 2\pi Rj} x(1-x)\left[K_0(z) - K_2(z)\right]\right\},$$

$$T^{\ell r} \equiv \frac{\beta_0 \lambda}{16\pi^2}\left\{\delta^{\ell r}\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) + 12\sum_{j\neq 0}(\sigma_{j,N})_{\ell r}\int_0^1 dx\, e^{ixp_3 2\pi Rj} x(1-x)\left[K_0(z) - \frac{p_3^2}{p^2}K_2(z)\right]\right\},$$

$$L \equiv \frac{\beta_0 \lambda}{16\pi^2}\left\{\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) + 12\sum_{j\neq 0, j=0 \bmod N}\int_0^1 dx\, e^{ixp_3 2\pi Rj} x(1-x)\left[K_0(z) - K_2(z)\right]\right\},$$

$$T \equiv \frac{\beta_0 \lambda}{16\pi^2}\left\{\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) + 12\sum_{j\neq 0, j=0 \bmod N}\int_0^1 dx\, e^{ixp_3 2\pi Rj} x(1-x)\left[K_0(z) - \frac{p_3^2}{p^2}K_2(z)\right]\right\},$$

$z \equiv \sqrt{x(1-x)p^2}2\pi R|j|$ and

$$\beta_0 = -\frac{2}{3}n_W$$

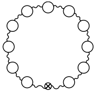
is the one-loop coefficient of the beta function.

- Then, to include the effect of the gauge field partially, we set by hand,

$$\beta_0 \to \frac{11}{3} - \frac{2}{3}n_W.$$

Gluon condensate in $N \to \infty$

- In the large β_0-approximation, the gluon condensate is computed as



That is,

$$
\begin{aligned}
&\left\langle \mathrm{tr}(\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu})\right\rangle \\
&= -\frac{1}{2}\left\langle(\partial_\mu\tilde{A}_\nu^\ell - \partial_\nu\tilde{A}_\mu^\ell)^2\right\rangle - \frac{1}{2}\left\langle(\partial_\mu\tilde{A}_\nu^{mn} - \partial_\nu\tilde{A}_\mu^{mn})(\partial_\mu\tilde{A}_\nu^{nm} - \partial_\nu\tilde{A}_\mu^{nm})\right\rangle \\
&= -\frac{\lambda}{N}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2\pi R}\sum_{p_3\in\mathbb{Z}/R}\sum_{\ell=1}^{N-1}\left\{\left[(1-L)^{-1}\right]^{\ell\ell} + 2\left[(1-T)^{-1}\right]^{\ell\ell}\right\} \\
&\quad -\frac{\lambda}{N}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2\pi R}\sum_{p_3\in\mathbb{Z}/R}\sum_{\substack{m\neq n\\ 1\leq m,n\leq N}}\left[(1-L)^{-1} + 2(1-T)^{-1}\right]_{p\to p_{mn}}.
\end{aligned}
$$

- We consider $N \to \infty$. The parts containing the Bessel functions are then suppressed.

Borel singularity in the gluon condensate

- Perturbative expansion $\left\langle \mathrm{tr}(\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu}) \right\rangle \sim \lambda \sum_{k=0}^{\infty} f_k \left(\frac{\beta_0 \lambda}{16\pi^2}\right)^k$.
- Contribution from the photon

$$(f_k)_{\text{photon}} = -3 \int \frac{d^3p}{(2\pi)^3} \underbrace{\int \frac{dp_3}{2\pi} \sum_{j=-\infty}^{\infty} e^{ip_3 2\pi Rj}}_{=1/(2\pi R) \sum_{p_3 \in \mathbb{Z}/R}} \left[\ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) \right]^k .$$

- The Borel transform (q is the UV cutoff)

$$\begin{aligned} B(u)_{\text{photon}} &= -3 \sum_{j=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} e^{ip_3 2\pi Rj} \left(\frac{e^{5/3}\mu^2}{p^2} \right)^u \\ &= \frac{3}{16\pi^2} (e^{5/3}\mu^2)^u \left[(q^2)^{2-u} \frac{1}{u-2} - 2(\pi^2 R^2)^{u-2} \frac{\Gamma(2-u)}{\Gamma(u)} \zeta(4-2u) \right] \\ &\overset{u \sim 3/2}{\sim} \frac{3}{16\pi^2} (e^{5/3}\mu^2)^{3/2} 2(\pi^2 R^2)^{-1/2} \frac{1}{u-3/2} . \end{aligned}$$

- Photon produces a singularity at $u = 3/2$, but not $u = 2$, similar to $\mathbb{C}P^{N-1}$.

Borel singularity in the gluon condensate

- Contribution of the W-boson,

$$(f_k)_{\text{W-boson}} = -3 \int \frac{d^4p}{(2\pi)^4} \sum_{j=-\infty}^{\infty} e^{ip_3 2\pi Rj} \frac{1}{N} \sum_{\substack{m\neq n \\ 1\leq m,n\leq N}} e^{i(m-n)2\pi j/N} \left[\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right]^k .$$

- Noting the relation,

$$\frac{1}{N} \sum_{\substack{m\neq n \\ 1\leq m,n\leq N}} e^{i(m-n)2\pi j/N} = \begin{cases} N-1, & \text{for } j=0 \mod N, \\ -1, & \text{for } j\neq 0 \mod N, \end{cases}$$

we have

$$(f_k)_{\text{W-boson}} = -3 \int \frac{d^4p}{(2\pi)^4} \sum_{j=-\infty}^{\infty} e^{ip_3 2\pi RNj} [(N-1)-(-1)] \left[\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right]^k$$

$$-3 \int \frac{d^4p}{(2\pi)^4} \sum_{j=-\infty}^{\infty} e^{ip_3 2\pi Rj} (-1) \left[\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right)\right]^k \Leftarrow \text{cancels photon}$$

Borel singularity in the gluon condensate

- Contribution of the W-boson is thus

$$(f_k)_{\text{W-boson}} = -3N \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi RN} \sum_{p_3 \in \mathbb{Z}/(RN)} \left[\ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) \right]^k - (f_k)_{\text{photon}}$$

- S^1 is de-compactified in $N \to \infty$! (cf. Eguchi–Kawai, Gross–Kitazawa; Sulejmanpasic (arXiv:1610.04009)) and, for $N \to \infty$,

$$(f_k)_{\text{W-boson}} = -3N \int \frac{d^4p}{(2\pi)^4} \left[\ln \left(\frac{e^{5/3}\mu^2}{p^2} \right) \right]^k - (f_k)_{\text{photon}}$$

- Borel transform is the 4D one and

$$\begin{aligned} B(u)_{\text{W-boson}} &= -3N \int \frac{d^4p}{(2\pi)^4} \left(\frac{e^{5/3}\mu^2}{p^2} \right)^u - B(u)_{\text{photon}}. \\ &= \frac{3N}{16\pi^2} (e^{5/3}\mu^2)^u (q^2)^{2-u} \frac{1}{u-2} - B(u)_{\text{photon}}. \end{aligned}$$

Summary

- In 4D QCD(adj.) on $\mathbb{R}^3 \times S^1$, in the large β_0-approximation, for $N \to \infty$, the Borel transform of the gluon condensate is given by

$$
\begin{aligned}
B(u) &= B(u)_{\text{photon}} + B(u)_{\text{W-boson}} \\
&= \frac{3N}{16\pi^2}(e^{5/3}\mu^2)^u (q^2)^{2-u} \frac{1}{u-2} \\
&\stackrel{u\sim 2}{\sim} \frac{3N}{16\pi^2}(e^{5/3}\mu^2)^2 \frac{1}{u-2}.
\end{aligned}
$$

- The Borel singularity at $u = 2$ and this is the same as $\mathbb{R}^4$.
- The situation is completely different from that in 2D $\mathbb{C}P^{N-1}$.
- For this, the contribution of the W-boson is crucial.
- If you do not like the UV divergence of the gluon condensate, we may consider the gradient flow (Lüscher) version, that is perfectly UV finite and exhibits the same Borel singularity.
- We should investigate how the situation changes as a function of N.
- Yes, we have no unified understanding yet. . .