

# Modular Invariance Approach to the Flavour Problem

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Understanding the origin of the pattern of neutrino mixing and of neutrino mass squared differences that emerged from the neutrino oscillation data in the recent years is one of the most challenging problems in neutrino physics. It is part of the more general fundamental problem in particle physics of understanding the origins of flavour in the quark and lepton sectors, i.e., of the patterns of quark masses and mixing, and of the charged lepton and neutrino masses and of neutrino mixing.

“Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesn't have to think for long: he wants to be able to explain the observed pattern of quark and lepton masses.”

**From Model Physicist, CERN Courier, 13 October 2017.**

Of fundamental importance are also

- the determination of the status of lepton charge conservation and the nature - Dirac or Majorana - of massive neutrinos (which is one of the most challenging and pressing problems in present day elementary particle physics) (GERDA, CUORE, KamLAND-Zen, EXO, LEGEND, nEXO,...);
- determining the status of CP symmetry in the lepton sector (T2K, NO $\nu$ A; T2HK, DUNE);
- determination of the type of spectrum neutrino masses possess, or the “neutrino mass ordering” (T2K + NO $\nu$ A; JUNO; PINGU, ORCA; T2HK, DUNE);
- determination of the absolute neutrino mass scale, or  $\min(m_j)$  (KATRIN, new ideas; cosmology).

The program of research extends beyond 2030.

- **BS3 $\nu$ RM: eV scale sterile  $\nu$ 's; NSI's; ChLFV processes ( $\mu \rightarrow e + \gamma$ ,  $\mu \rightarrow 3e$ ,  $\mu^- - e^-$  conversion on  $(A, Z)$ );  $\nu$ -related BSM physics at the TeV scale ( $N_{jR}$ ,  $H^{--}$ ,  $H^-$ , etc.).**

## Reference Model: 3- $\nu$ mixing

$$\nu_{lL} = \sum_{j=1}^3 U_{lj} \nu_{jL} \quad l = e, \mu, \tau.$$

The PMNS matrix  $U$  -  $3 \times 3$  unitary.

$\nu_j, m_j \neq 0$ : Dirac or Majorana particles.

Data: 3  $\nu$ s are light:  $\nu_{1,2,3}, m_{1,2,3} \lesssim 0.5$  eV.

3- $\nu$  mixing: 3-flavour neutrino oscillations possible.

$\nu_\mu, E$ ; at distance  $L$ :  $P(\nu_\mu \rightarrow \nu_{\tau(e)}) \neq 0, P(\nu_\mu \rightarrow \nu_\mu) < 1$

$P(\nu_l \rightarrow \nu_{l'}) = P(\nu_l \rightarrow \nu_{l'}; E, L; U; m_2^2 - m_1^2, m_3^2 - m_1^2)$

# Three Neutrino Mixing

$$\nu_{lL} = \sum_{j=1}^3 U_{lj} \nu_{jL} .$$

$U$  is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix,

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix}$$

- $U$  -  $n \times n$  unitary:

	$n$	2	3	4
mixing angles:	$\frac{1}{2}n(n-1)$	1	3	6

CP-violating phases:

• $\nu_j$ - Dirac:	$\frac{1}{2}(n-1)(n-2)$	0	1	3
• $\nu_j$ - Majorana:	$\frac{1}{2}n(n-1)$	1	3	6

$n = 3$ : 1 Dirac and  
2 additional CP-violating phases, Majorana phases

S.M. Bilenky, J. Hosek, S.T.P., 1980

# PMNS Matrix: Standard Parametrization

$$U = VP, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix},$$

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

- $s_{ij} \equiv \sin \theta_{ij}$ ,  $c_{ij} \equiv \cos \theta_{ij}$ ,  $\theta_{ij} = [0, \frac{\pi}{2}]$ ,
- $\delta$  - Dirac CPV phase,  $\delta = [0, 2\pi]$ ; CP inv.:  $\delta = 0, \pi, 2\pi$ ;
- $\alpha_{21}, \alpha_{31}$  - Majorana CPV phases; CP inv.:  $\alpha_{21(31)} = k(k')\pi$ ,  $k(k') = 0, 1, 2, \dots$   
S.M. Bilenky et al., 1980
- $\Delta m_{\odot}^2 \equiv \Delta m_{21}^2 \cong 7.37 \times 10^{-5} \text{ eV}^2 > 0$ ,  $\sin^2 \theta_{12} \cong 0.297$ ,  $\cos 2\theta_{12} \gtrsim 0.29$  ( $3\sigma$ ),
- $|\Delta m_{31(32)}^2| \cong 2.53$  (2.43) [2.56 (2.54)]  $\times 10^{-3} \text{ eV}^2$ ,  $\sin^2 \theta_{23} \cong 0.437$  (0.569) [0.425 (0.589)], NO (IO),
- $\theta_{13}$  - the CHOOZ angle:  $\sin^2 \theta_{13} = 0.0214$  (0.0218) [0.0215 (0.0216)], NO (IO).

F. Capozzi et al. (Bari Group), arXiv:1601.07777 [arXiv:1703.04471].

- $\text{sgn}(\Delta m_{\text{atm}}^2) = \text{sgn}(\Delta m_{31(32)}^2)$  not determined

$$\Delta m_{\text{atm}}^2 \equiv \Delta m_{31}^2 > 0, \quad \text{normal mass ordering (NO)}$$

$$\Delta m_{\text{atm}}^2 \equiv \Delta m_{32}^2 < 0, \quad \text{inverted mass ordering (IO)}$$

Convention:  $m_1 < m_2 < m_3$  - NO,  $m_3 < m_1 < m_2$  - IO

$$\Delta m_{31}^2(\text{NO}) = -\Delta m_{32}^2(\text{IO})$$

$$m_1 \ll m_2 < m_3, \quad \text{NH,}$$

$$m_3 \ll m_1 < m_2, \quad \text{IH,}$$

$$m_1 \cong m_2 \cong m_3, \quad m_{1,2,3}^2 \gg |\Delta m_{31(32)}^2|, \quad \text{QD; } m_j \gtrsim 0.10 \text{ eV.}$$

- $m_2 = \sqrt{m_1^2 + \Delta m_{21}^2}, \quad m_3 = \sqrt{m_1^2 + \Delta m_{31}^2}$  - NO;
- $m_1 = \sqrt{m_3^2 + \Delta m_{23}^2 - \Delta m_{21}^2}, \quad m_2 = \sqrt{m_3^2 + \Delta m_{23}^2}$  - IO;



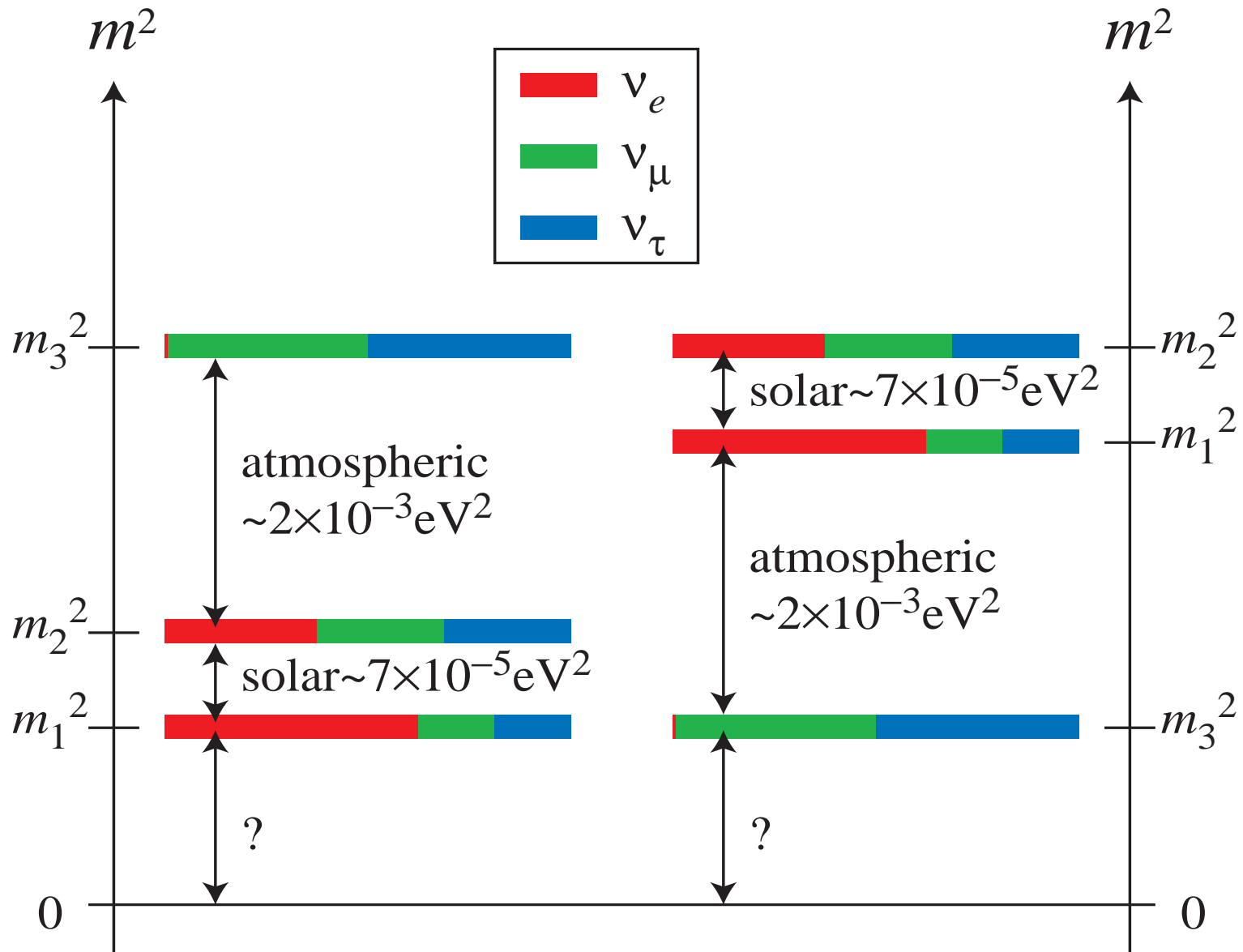


Table 3: Best fit values and allowed ranges at  $N\sigma = 1, 2, 3$  for the  $3\nu$  oscillation parameters, in either NO or IO. The latter column shows the formal “ $1\sigma$  accuracy” for each parameter, defined as  $1/6$  of the  $3\sigma$  range divided by the best-fit value (in percent).

Parameter	Ordering	Best fit	$1\sigma$ range	$2\sigma$ range	$3\sigma$ range	“ $1\sigma$ ” (%)
$\Delta m_{\odot}^2/10^{-5} \text{ eV}^2$	NO	7.34	7.20 – 7.51	7.05 – 7.69	6.92 – 7.91	2.2
	IO	7.34	7.20 – 7.51	7.05 – 7.69	6.92 – 7.91	2.2
$ \Delta m_{\text{A}}^2 /10^{-3} \text{ eV}^2$	NO	2.49	2.46 – 2.53	2.43 – 2.56	2.39 – 2.59	1.4
	IO	2.48	2.44 – 2.51	2.41 – 2.54	2.38 – 2.58	1.4
$\sin^2 \theta_{12}$	NO	3.04	2.91 – 3.18	2.78 – 3.32	2.65 – 3.46	4.4
	IO	3.03	2.90 – 3.17	2.77 – 3.31	2.64 – 3.45	4.4
$\sin^2 \theta_{13}/10^{-2}$	NO	2.14	2.07 – 2.23	1.98 – 2.31	1.90 – 2.39	3.8
	IO	2.18	2.11 – 2.26	2.02 – 2.35	1.95 – 2.43	3.7
$\sin^2 \theta_{23}/10^{-1}$	NO	5.51	4.81 – 5.70	4.48 – 5.88	4.30 – 6.02	5.2
	IO	5.57	5.33 – 5.74	4.86 – 5.89	4.44 – 6.03	4.8
$\delta/\pi$	NO	1.32	1.14 – 1.55	0.98 – 1.79	0.83 – 1.99	14.6
	IO	1.52	1.37 – 1.66	1.22 – 1.79	1.07 – 1.92	9.3

$$\Delta m_{\odot}^2 \equiv \Delta m_{21}^2; \quad \Delta m_{\text{A}}^2 \equiv \Delta m_{31(32)}^2, \quad \text{NO (IO)}.$$

F. Capozzi et al. (Bari Group), arXiv:1804.09678.

- Dirac phase  $\delta$ :  $\nu_l \leftrightarrow \nu_{l'}$ ,  $\bar{\nu}_l \leftrightarrow \bar{\nu}_{l'}$ ,  $l \neq l'$ ;  $A_{CP}^{(l,l')} \propto J_{CP} \propto \sin \theta_{13} \sin \delta$ :

P.I. Krastev, S.T.P., 1988

$$J_{CP} = \text{Im} \{ U_{e1} U_{\mu 2} U_{e2}^* U_{\mu 1}^* \} = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta$$

Current data:  $|J_{CP}| \lesssim 0.035$  (can be relatively large!); b.f.v. with  $\delta = 3\pi/2$ :  
 $J_{CP} \cong -0.035$ .

- Majorana phases  $\alpha_{21}$ ,  $\alpha_{31}$ :

–  $\nu_l \leftrightarrow \nu_{l'}$ ,  $\bar{\nu}_l \leftrightarrow \bar{\nu}_{l'}$  not sensitive;

S.M. Bilenky et al., 1980;  
P. Langacker et al., 1987

–  $|\langle m \rangle|$  in  $(\beta\beta)_{0\nu}$ -decay depends on  $\alpha_{21}$ ,  $\alpha_{31}$ ;

–  $\Gamma(\mu \rightarrow e + \gamma)$  etc. in SUSY theories depend on  $\alpha_{21,31}$ ;

– BAU, leptogenesis scenario:  $\delta, \alpha_{21,31}$  !

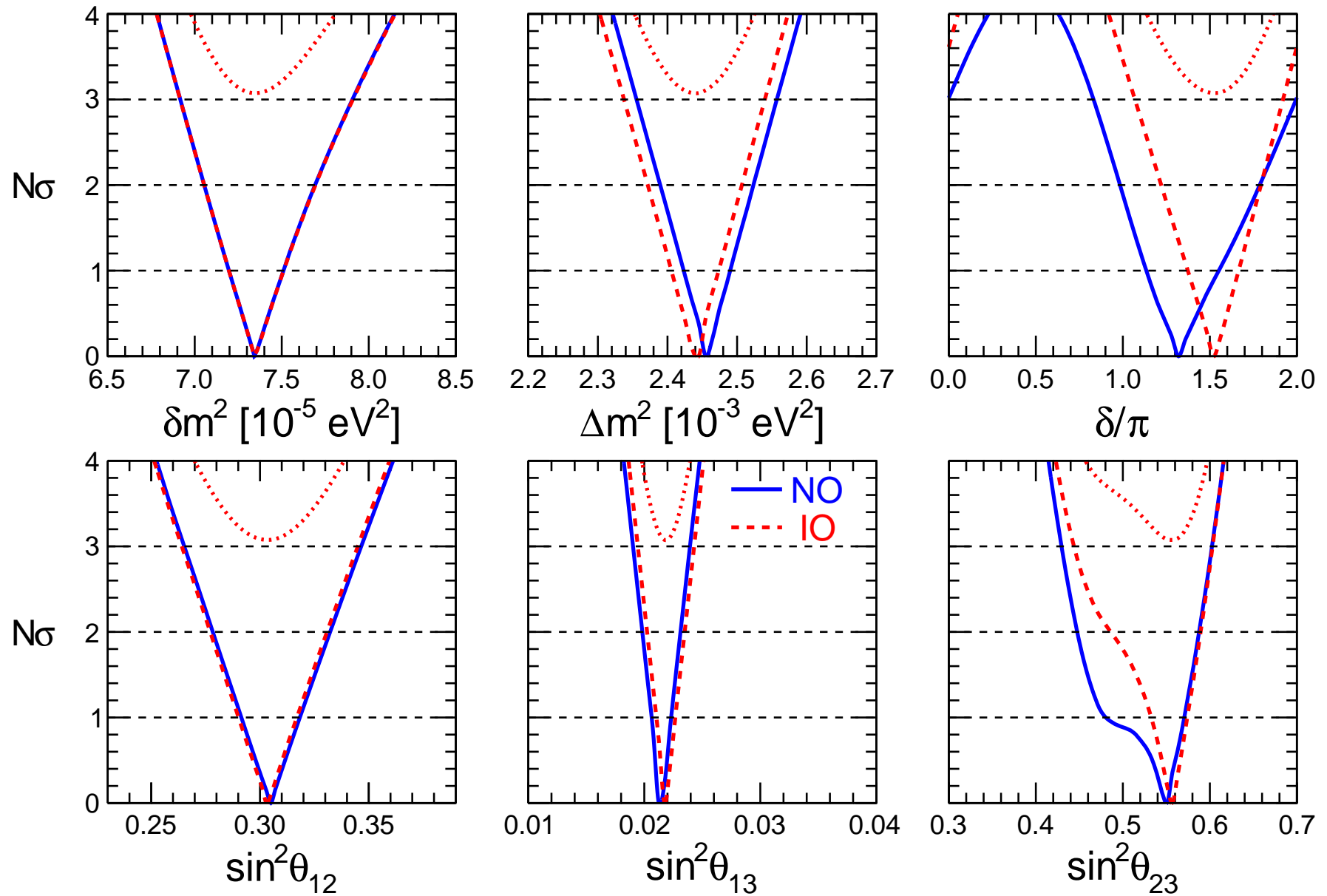
$$\delta \cong 3\pi/2?$$

$$\begin{aligned} J_{CP} &= \text{Im} \{ U_{e1} U_{\mu 2} U_{e2}^* U_{\mu 1}^* \} \\ &= \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta \end{aligned}$$

- **Best fit value:**  $\delta = 1.32 (1.52)\pi$  [ $1.30 (1.54)\pi$ ];
- $\delta = 0$  or  $2\pi$  are disfavored at  $3.0 (3.6)\sigma$  [ $2.6 (3.0)\sigma$ ];
- $\delta = \pi$  is disfavored at  $1.8 (3.6)\sigma$  [ $1.7 (3.3)\sigma$ ];
- $\delta = \pi/2$  is strongly disfavored at  $4.4 (5.2)\sigma$  [ $4.3 (5.0)\sigma$ ].
- **At  $3\sigma$ :**  $\delta/\pi$  is found to lie in **0.83-1.99 (1.07-1.92)** [**1.07-1.97 (0.80-2.08)**].

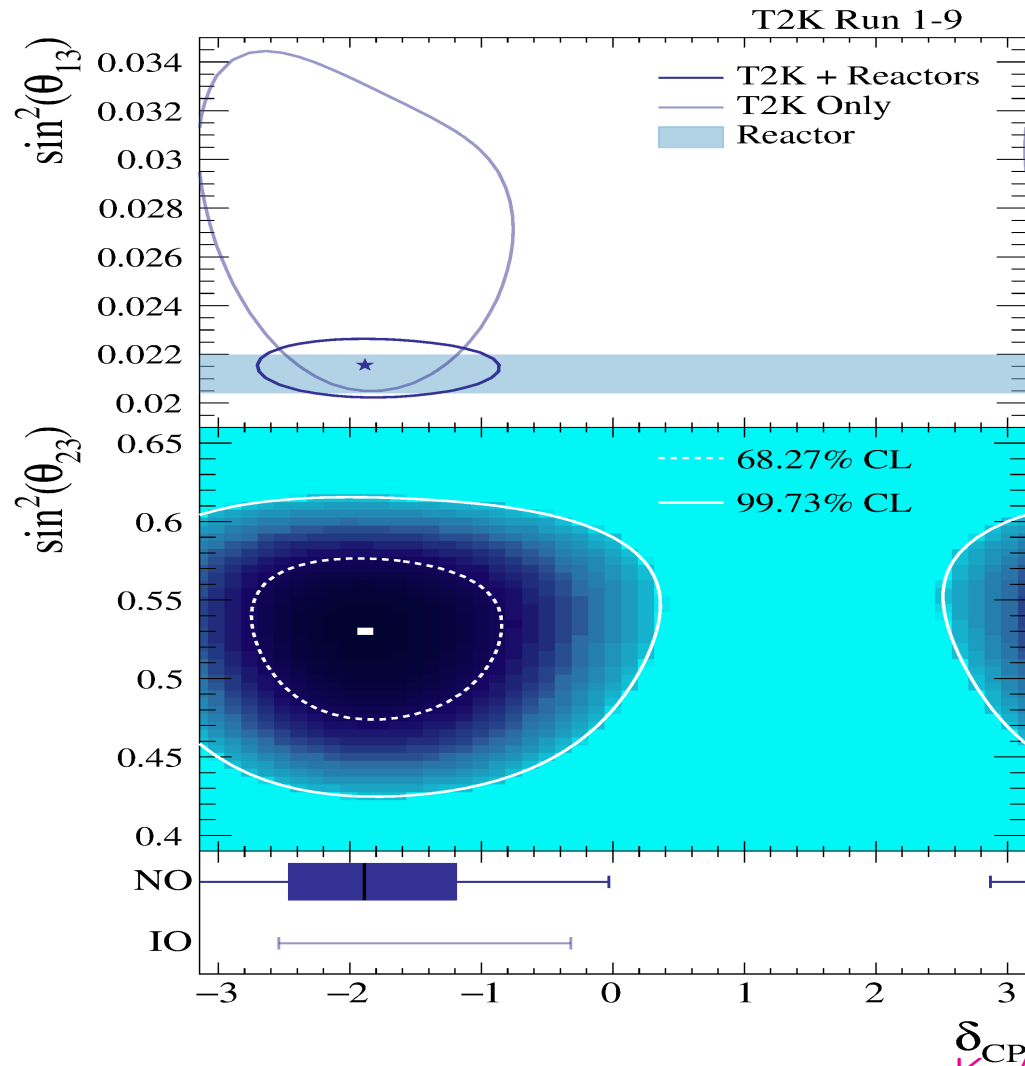
F. Capozzi, E. Lisi *et al.*, arXiv:1804.09678 [E. Esteban *et al.*, NuFit 3.2 (Jan., 2018)]

LBL Acc + Solar + KamLAND + SBL Reactors + Atmos



F. Capozzi et al. (Bari Group), arXiv:1804.09678.

# Latest results from T2K



K. Abe et al., 1910.03887

**Best fit value:  $\delta = -1.89$  ( $-1.38$ ), NO (IO).**

**$\delta = 0, \pi$  ruled out at 95% CL.**

**At  $3\sigma$ :  $\delta$  is found to lie in  $[-3.41, -0.03]$  ( $[-2.54, -0.32]$ ), NO (IO).**

**Latest global analysis: data favors NO**

**IO disfavored at  $3.1\sigma$ .**

F. Capozzi et al., 1804.09678.



# Quark Masses and Mixing

The observed patterns of the masses of up- and down-type quarks and of the charged leptons of the three families of SM are characterised by strong hierarchies:

$$m_d \ll m_s \ll m_b, \quad \frac{m_d}{m_s} = 5.02 \times 10^{-2}, \quad \frac{m_s}{m_b} = 2.22 \times 10^{-2}, \quad m_b = 4.18 \text{ GeV};$$

$$m_u \ll m_c \ll m_t, \quad \frac{m_u}{m_c} = 1.7 \times 10^{-3}, \quad \frac{m_c}{m_t} = 7.3 \times 10^{-3}, \quad m_t = 172.9 \text{ GeV};$$

$$m_e \ll m_\mu \ll m_\tau, \quad \frac{m_e}{m_\mu} = 4.8 \times 10^{-3}, \quad \frac{m_\mu}{m_\tau} = 5.95 \times 10^{-2}, \quad m_\tau = 1776.86 \text{ MeV}.$$

The three quark mixing angles are small and hierarchical,

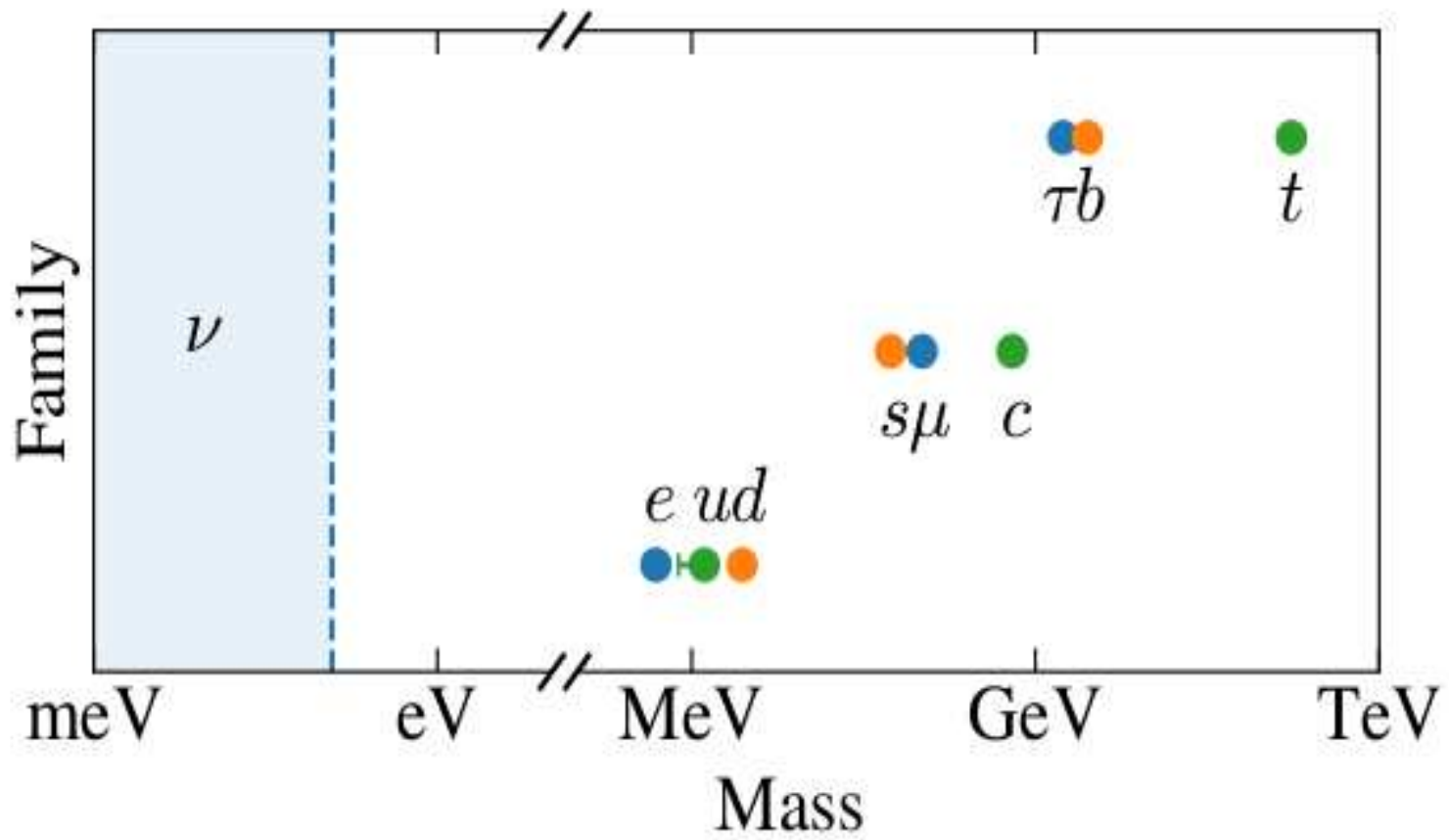
$$\theta_{12}^q = 12.96^\circ, \quad \theta_{23}^q = 2.42^\circ, \quad \theta_{13}^q = 0.022^\circ,$$

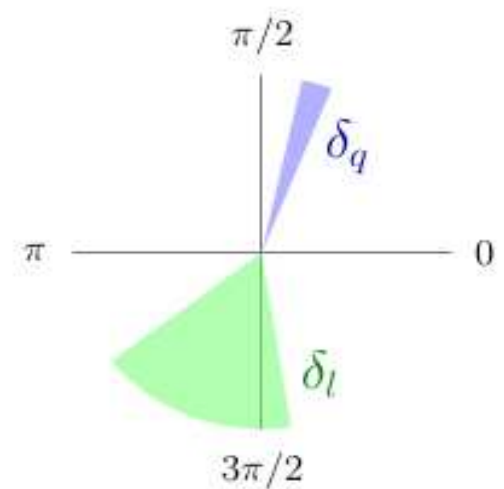
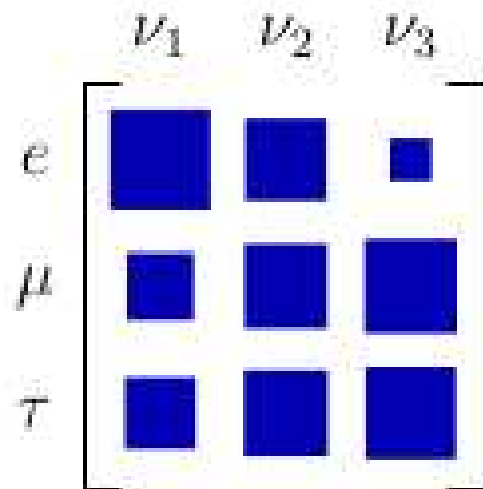
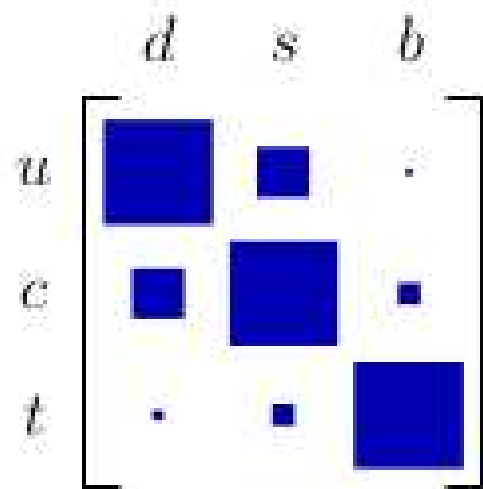
while the lepton mixing is characterised by two large and one small angles,

$$\theta_{12}^l = 33.65^\circ, \quad \theta_{23}^l = 47.1^\circ, \quad \theta_{13}^l = 8.49^\circ.$$

The quoted values correspond to the standard" parametrisations of  $V_{\text{CKM}}$  and  $U_{\text{PMNS}}$ . The Dirac CPV phases in CKM and PMNS matrices read:

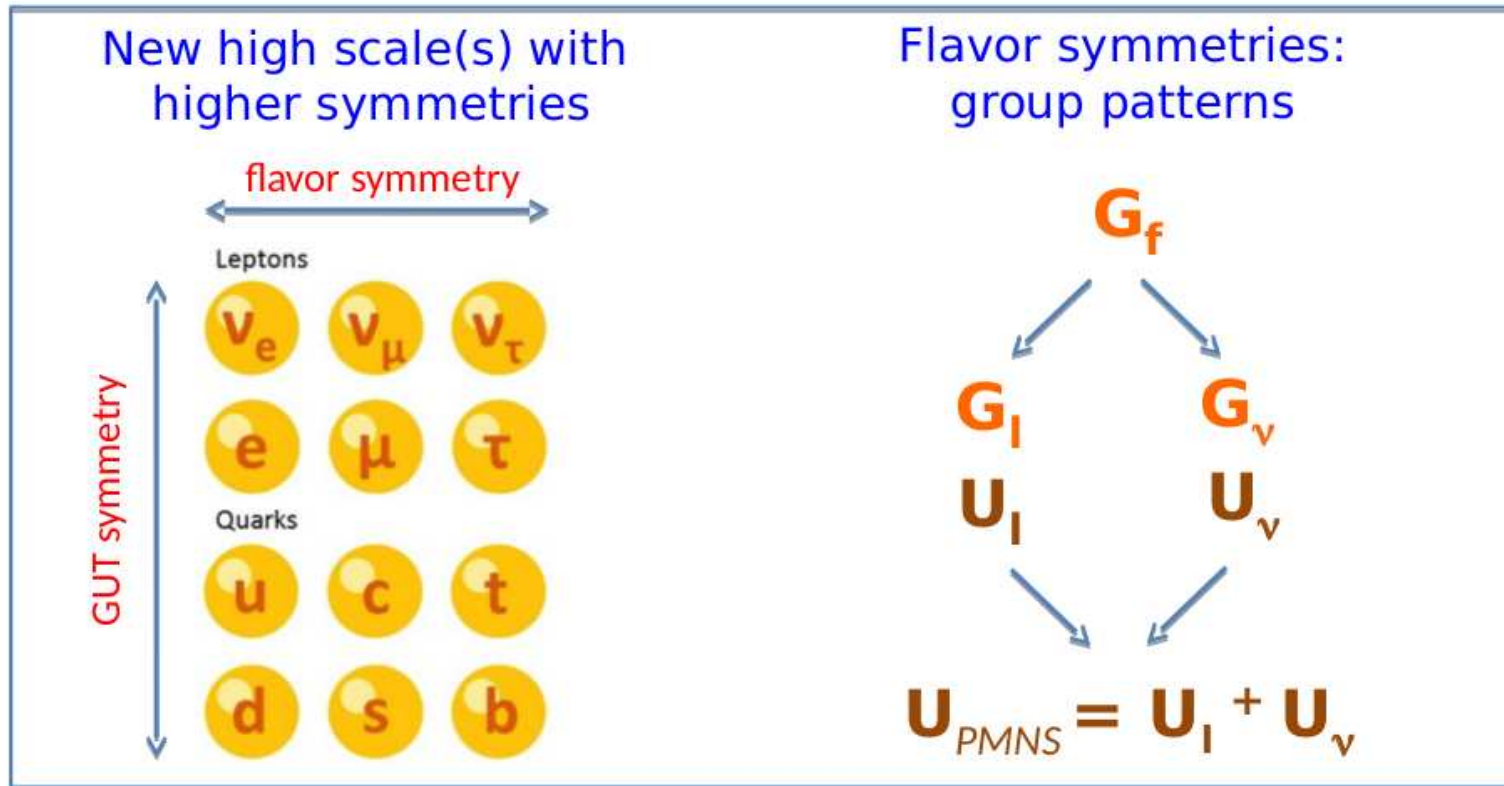
$$\delta_q = (73.5 - 5.1 + 4.2)^\circ, \quad \delta_l = (1.37 - 0.16 + 0.18) \times 180^\circ.$$





# The Flavour Problem

## Model building with symmetries



E. Lisi, TAUP 2019

# Modular Invariance

Modular invariance has been investigated in the context of field and superstring theories, being a feature of a number of theoretical physics constructions (theories with extra dimensions compactified on a torus (or tori), superstring theories on tori or orbifolds, supergravity theories) [2]-[7]; it can be present in theories with global or local super-symmetry and appears to be a property of the quantum Hall effect [8]-[13]. The modular forms which are an integral part of the approach (see below) have been extensively studied by mathematicians, in particular, in connection with number theory [14].

[2] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, Phys. Rept. 445, 1 (2007). [3] L. E. Ibanez, Phys. Lett. B181, 269 (1986). [4] S. Hamidi and C. Vafa, Nucl. Phys. B279, 465 (1987). [5] S. Ferrara, D. Lust and S. Theisen, Phys. Lett. B233, 147 (1989). [6] D. Cremades, L. E. Ibanez and F. Marchesano, JHEP 0405, 079 (2004). [7] S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B225, 363 (1989). [8] C. A. Lutken and G. G. Ross, Phys. Rev. D45, 11837 (1992). [9] A. Cappelli and G. R. Zemba, Nucl. Phys. B490, 595 (1997). [10] C. P. Burgess and B. P. Dolan, Phys. Rev. B63, 155309 (2001). [11] M. Lippert, R. Meyer and A. Taliotis, JHEP 1501, 023 (2015). [12] C.A. Lutken, EPJ Web Conf. 71, 0079 (2014) 00079 (doi:10.1051/epjconf/20147100079). [13] C. A. Lutken, Phys. Rev. B99, 195152 (2019). [14] H. M. Farkas and I. Kra, Theta Constants, Riemann Surfaces and the Modular Group, Graduate Studies in Mathematics, vol. 37, American Mathematical Society (2001).

# The Modular Group and the Finite Modular Groups

The modular group  $\bar{\Gamma}$  – group of linear fractional transformations  $\gamma$  acting on the complex variable  $\tau$  belonging to the upper-half complex plane:

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad \text{Im}\tau > 0.$$

$\bar{\Gamma}$  is generated by two transformations  $S$  and  $T$  satisfying

$$S^2 = (ST)^3 = I,$$

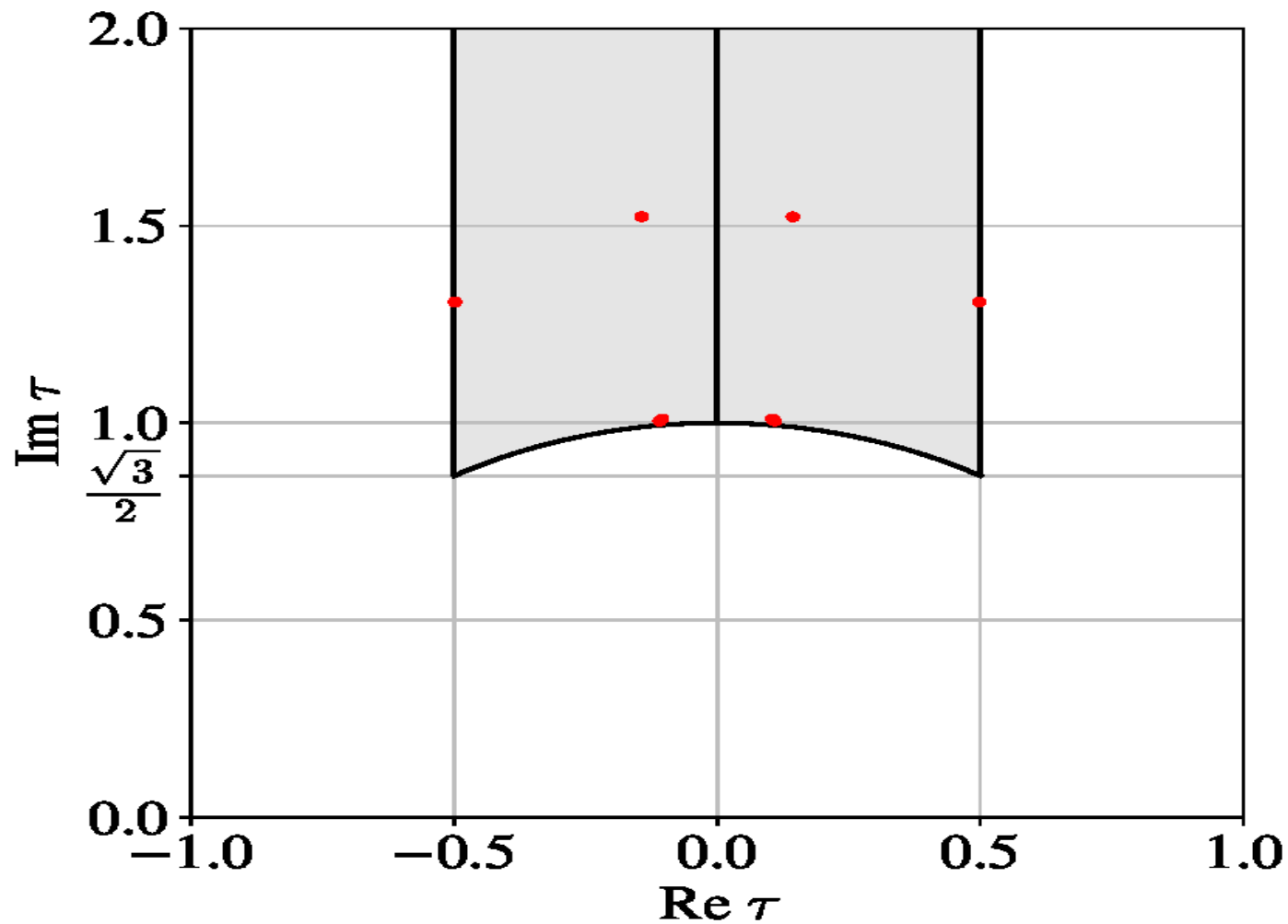
$I$  being the identity element, and acting on  $\tau$  as

$$\tau \xrightarrow{S} -\frac{1}{\tau}, \quad \tau \xrightarrow{T} \tau + 1.$$

$S$  and  $T$  can be represented as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

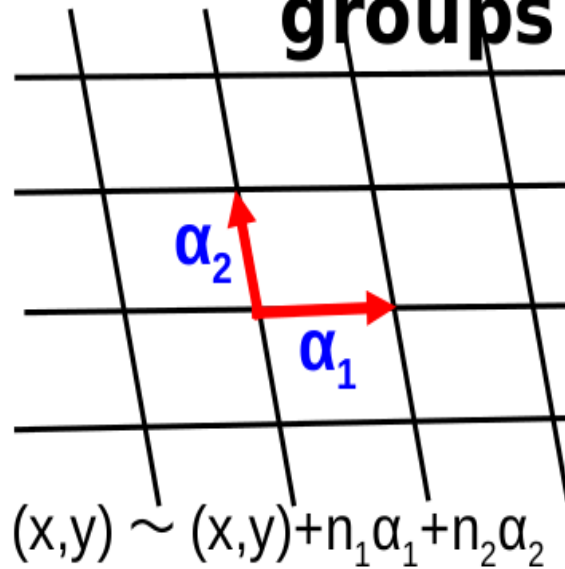
**Complex variable  $\tau$  - modulus.**



The Fundamental Domain of  $\bar{\Gamma}$  shown for  $\text{Im } \tau \leq 2$  (the red dots correspond to solutions of the lepton flavour problem, see further).

P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933.

# Modular groups



## Toroidal compactification

Two-dimensional torus  $T^2$  is obtained as

$$T^2 = \mathbb{R}^2 / \Lambda$$

$\Lambda$  is two-dimensional lattice, which is spanned by two lattice vectors

$$\alpha_1 = 2\pi R \text{ and } \alpha_2 = 2\pi R\tau$$

$$(x, y) \sim (x, y) + n_1 \alpha_1 + n_2 \alpha_2$$

$\tau = \alpha_2 / \alpha_1$  is a complex modulus parameter.

The same lattice can be spanned by other bases.

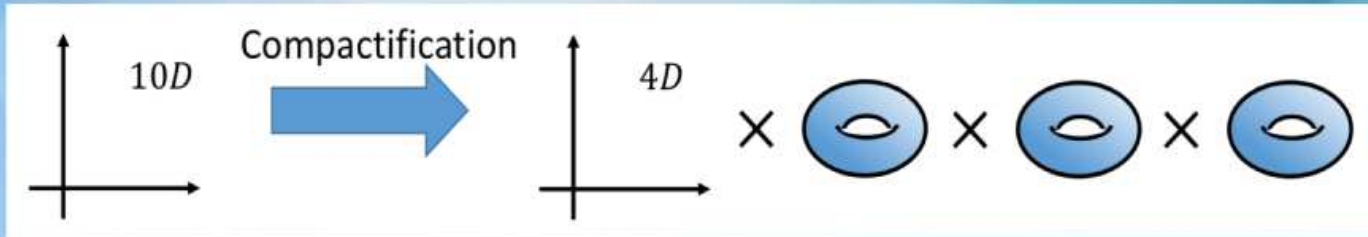
$$\begin{pmatrix} \alpha'_2 \\ \alpha'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix} \quad \begin{array}{l} \mathbf{ad-bc=1} \\ \mathbf{a,b,c,d \text{ are integer}} \end{array} \quad \mathbf{SL(2,Z)}$$

[Slide from M. Tanimoto's talk at SISSA, June 11, 2018]

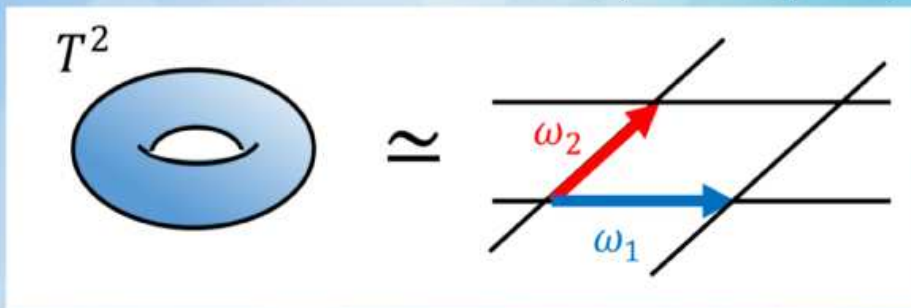


# Modular Invariance as Flavour Symmetry

string theory in  $d=10$  need 6 compact dimensions



simplest compactification: 3 copies of a torus  $T^2$



completely characterized by

$$\tau = \frac{\omega_2}{\omega_1} \quad \text{Im}(\tau) > 0$$

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(g(\tau), \varphi)$$

lattice left invariant by modular transformations:

$$\tau \rightarrow \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}$$

$a, b, c, d$  integers  
 $ad - bc = 1$



$\mathcal{L}_{eff}$  modular invariant

[figures from Tatsuishi's talk at Flasy 2018]

$\bar{\Gamma}$  is isomorphic to the projective special linear group  $PSL(2, Z) = SL(2, Z)/Z_2$ ,  $SL(2, Z)$  is the special linear group of  $2 \times 2$  matrices with integer elements and unit determinant, and  $Z_2 = \{I, -I\}$  is its centre.

$SL(2, Z) = \Gamma(1) \equiv \Gamma$  contains a series of infinite normal subgroups  $\Gamma(N)$ ,

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \quad N = 1, 2, 3, \dots,$$

called the principal congruence subgroups. For  $N = 1$  and  $2$ , we define the groups  $\bar{\Gamma}(N) \equiv \Gamma(N)/\{I, -I\}$  with  $\bar{\Gamma}(1) \equiv \bar{\Gamma}$ . For  $N > 2$ ,  $\bar{\Gamma}(N) \equiv \Gamma(N)$  since  $\Gamma(N)$  does not contain the subgroup  $\{I, -I\}$ .

The quotient groups  $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$  are called finite modular groups. Remarkably, for  $N \leq 5$ ,  $\Gamma_N$  are isomorphic to non-Abelian discrete groups widely used in flavour model building:

$\Gamma_2 \simeq S_3$ ,  $\Gamma_3 \simeq A_4$ ,  $\Gamma_4 \simeq S_4$  and  $\Gamma_5 \simeq A_5$ .

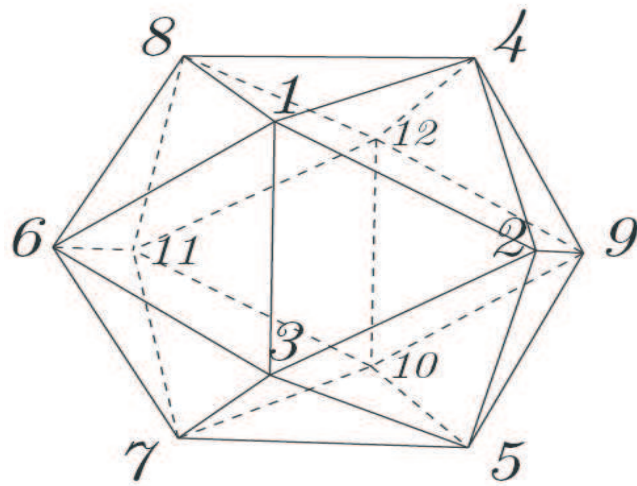
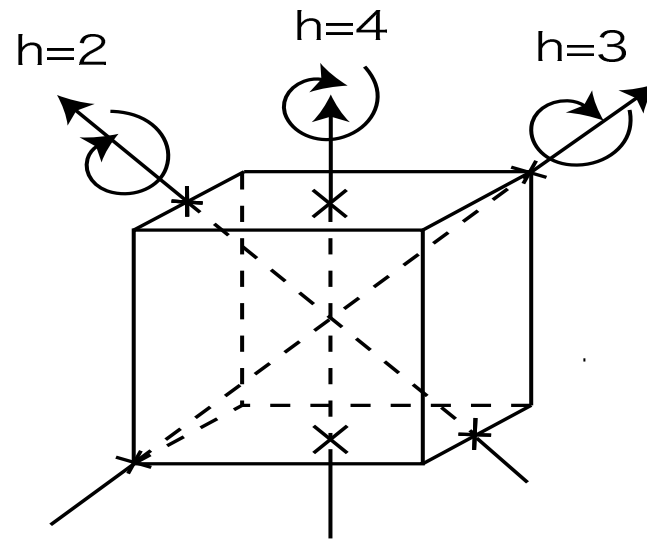
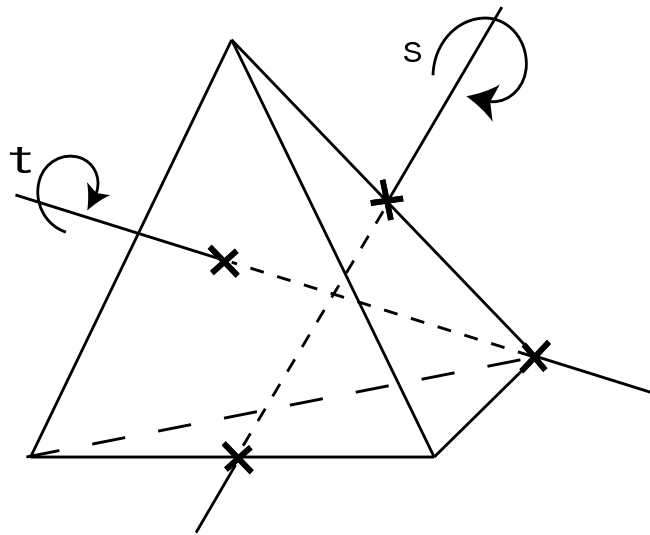
$\Gamma_N$  is presented by two generators  $S$  and  $T$  satisfying:

$$S^2 = (ST)^3 = T^N = I.$$

The group theory of  $\Gamma_2 \simeq S_3$ ,  $\Gamma_3 \simeq A_4$ ,  $\Gamma_4 \simeq S_4$  and  $\Gamma_5 \simeq A_5$  is summarised, e.g., in P.P. Novichkov *et al.*, **JHEP 07 (2019) 165**, arXiv:1905.11970.

Group	Number of elements	Generators	Irreducible representations
$S_4$	24	$S, T (U)$	$1, 1', 2, 3, 3'$
$A_4$	12	$S, T$	$1, 1', 1'', 3$
$T'$	24	$S, T (R)$	$1, 1', 1'', 2, 2', 2'', 3$
$A_5$	60	$\tilde{S}, \tilde{T}$	$1, 3, 3', 4, 5$

**Number of elements, generators and irreducible representations of  $S_4$ ,  $A_4$ ,  $T'$  and  $A_5$  discrete groups.**



Examples of symmetries:  $A_4$ ,  $S_4$ ,  $A_5$ .

From M. Tanimoto et al., arXiv:1003.3552

# Modular Forms

The key elements of the considered framework are modular forms  $f(\tau)$  of weight  $k$  and level  $N$  – holomorphic functions of  $\tau$ , which transform under  $\bar{\Gamma}(N)$  as follows:

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \bar{\Gamma}(N),$$

where the weight  $k \geq 0$  is an even number, level  $N$  is a natural number. For given  $k, N$ , the modular forms span a linear space of finite dimension. The dimension of the linear space of modular forms of weight  $k$  and level 3,  $\mathcal{M}_k(\Gamma_3 \simeq A_4)$ , is  $k + 1$ ;  
of weight  $k$  and level 4,  $\mathcal{M}_k(\Gamma_4 \simeq S_4)$ , is  $2k + 1$ ;  
of weight  $k$  and level 5,  $\mathcal{M}_k(\Gamma_5 \simeq A_5)$ , is  $5k + 1$ .  
Thus,  $\dim \mathcal{M}_2(\Gamma_3 \simeq A_4) = 3$ ,  $\dim \mathcal{M}_2(\Gamma_4 \simeq S_4) = 5$ ,  $\dim \mathcal{M}_2(\Gamma_5 \simeq A_5) = 11$ .  
One can find a basis  $F(\tau) \equiv (f_1(\tau), f_2(\tau), \dots)^T$  in each of these spaces such that for any  $\gamma \in \bar{\Gamma}$ ,  $F(\gamma\tau)$  belongs to the same space and transforms according to a unitary irreducible representation  $\rho$  of  $\Gamma_N$ :

$$F(\gamma\tau) = (c\tau + d)^k \rho(\gamma) F(\tau), \quad \gamma \in \bar{\Gamma}.$$

**This result is at the basis of the modular invariance approach to the flavour problem proposed in F. Feruglio, arXiv:1706.08749.**

It is of crucial importance to find the basis of modular forms of weight 2 transforming in irreps of  $\Gamma_N$ .

Multiplets of  $\Gamma_N$  of higher weight modular forms can be constructed from tensor products of the lowest weight 2 multiplets (they represent homogeneous polynomials of the weight 2 modular forms).

For ( $\Gamma_3 \simeq A_4$ ), the generating (basis) modular forms of weight 2 were shown to form a 3 of  $A_4$  (expressed in terms of the Dedekind eta function).

F. Feruglio, arXiv:1706.08749

For ( $\Gamma_4 \simeq S_4$ ), the 5 basis modular forms of weight 2 were shown to form a 2 and a 3' of  $S_4$  (expressed in terms of the Dedekind eta function).

J. Penedo, STP, arXiv:1806.11040

For ( $\Gamma_5 \simeq A_5$ ), the 11 basis modular forms of weight 2 were shown to form a 3, a 3' and a 5 of  $A_5$  (expressed in terms of the Jacobi theta function).

P.P. Novichkov, J. Penedo, STP, A.V. Titov, arXiv:1812.02158

For ( $\Gamma_2 \simeq S_3$ ), the 2 basis modular forms of weight 2 were shown to form a 2 of  $S_3$  (expressed in terms of the Dedekind eta function).

T. Kobayashi, K. Tanaka, T.H. Tatsuishi, arXiv:1803.10391

Multiplets of higher weight modular forms have been also constructed from tensor products of the lowest weight 2 multiplets:

i) for  $N = 4$  (i.e.,  $S_4$ ), multiplets of weight 4 (weight  $k \leq 10$ ) were derived in arXiv:1806.11040 (arXiv:1811.04933);

ii) for  $N = 3$  (i.e.,  $A_4$ ) multiplets of weight  $k \leq 6$  were found in arXiv:1706.08749;

iii) for  $N = 5$  (i.e.,  $A_5$ ), multiplets of weight  $k \leq 10$  were derived in arXiv:1812.02158.

The modular forms of level  $N = 2, 3, 4$  for  $\Gamma_{2,3,4} \simeq S_3, A_4, S_4$  have been constructed by use of the Dedekind eta function,  $\eta(\tau)$ ,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{i2\pi\tau}.$$

For  $A_4$ ,  $\eta(3\tau)$ ,  $\eta(\tau/3)$ ,  $\eta((\tau + 1)/3)$  and  $\eta((\tau + 2)/3)$  were used.

F. Feruglio, arXiv:1706.08749

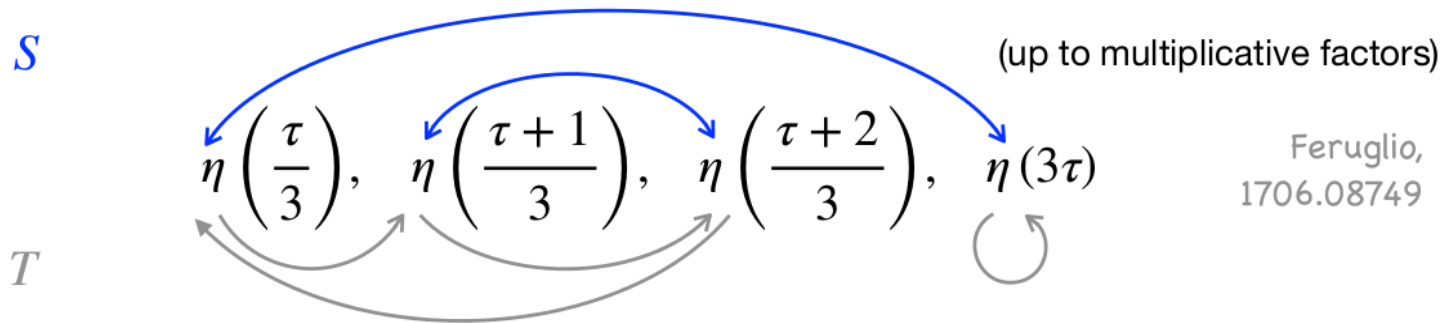
For  $S_4$ ,  $\eta(\tau + 1/2)$ ,  $\eta(4\tau)$ ,  $\eta(\tau/4)$ ,  $\eta((\tau + 1)/4)$ ,  $\eta((\tau + 2)/4)$  and  $\eta((\tau + 3)/4)$  were used.

J.T. Penedo, STP, arXiv:1806.11040

# Modular forms of weight 2

**Level  $N = 3$**  ( $\Gamma_3 \simeq A_4 : S^2 = (ST)^3 = T^3 = I$ )

Nk	0	2	4	6
3	1	3	5	7

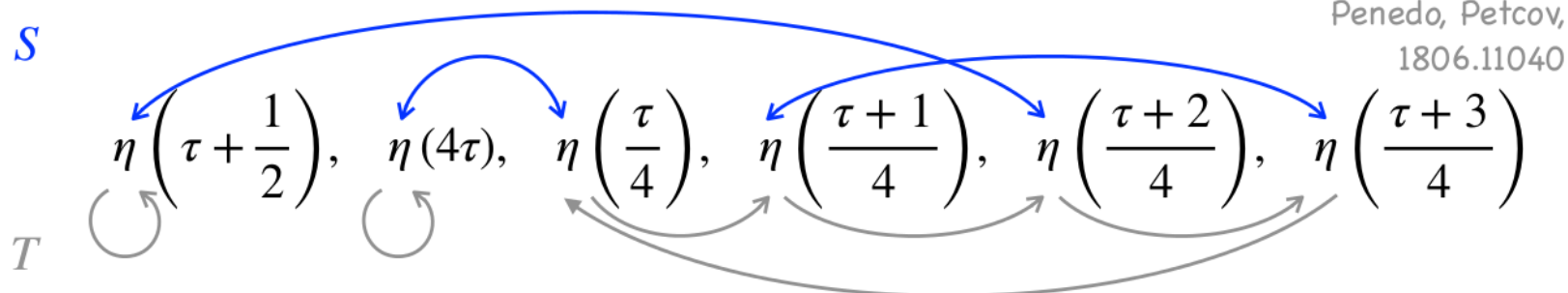


Feruglio,  
1706.08749

$A_4$  triplet of weight 2 modular forms

**Level  $N = 4$**  ( $\Gamma_4 \simeq S_4 : S^2 = (ST)^3 = T^4 = I$ )

Nk	0	2	4	6
4	1	5	9	13



Penedo, Petcov,  
1806.11040

$S_4$  doublet and triplet ( $3'$ ) of weight 2 modular forms

From A. Titov, talk at FLASY 2019



The modular forms of the weight 2 are expressed in terms of

$\eta'(\tau + 1/2)/\eta(\tau + 1/2)$ ,  $\eta'(4\tau)/\eta(4\tau)$ ,  $\eta'(\tau/4)/\eta(\tau/4)$ ,  $\eta'((\tau + 1)/4)/\eta((\tau + 1)/4)$ ,  
 $\eta'((\tau + 2)/4)/\eta((\tau + 2)/4)$ ,  $\eta'((\tau + 3)/4)/\eta((\tau + 3)/4)$ , and are written as:

$$Y_1(\tau) = Y(1, 1, \omega, \omega^2, \omega, \omega^2 | \tau),$$

$$Y_2(\tau) = Y(1, 1, \omega^2, \omega, \omega^2, \omega | \tau),$$

$$Y_3(\tau) = Y(1, -1, -1, -1, 1, 1 | \tau),$$

$$Y_4(\tau) = Y(1, -1, -\omega^2, -\omega, \omega^2, \omega | \tau),$$

$$Y_5(\tau) = Y(1, -1, -\omega, -\omega^2, \omega, \omega^2 | \tau), \quad \omega = e^{i2\pi/3}.$$

J.T. Penedo, STP, arXiv:1806.11040

$Y(a_1, a_2, a_3, a_4, a_5, a_6|\tau)$  is given by:

$$Y(a_1, a_2, a_3, a_4, a_5, a_6|\tau) = a_1 \frac{\eta'(\tau + 1/2)}{\eta(\tau + 1/2)} + 4a_2 \frac{\eta'(4\tau)}{\eta(4\tau)} + \frac{1}{4} \sum_{m=0}^3 a_{m+3} \frac{\eta'((\tau + m)/4)}{\eta((\tau + m)/4)}$$

where  $\sum_{i=1}^6 a_i = 0$ . The five independent modular form combinations in preceding equations are decomposed into the 2 and 3' irreducible representations of  $S_4$ :

$$Y_2^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \quad Y_{3'}^{(2)}(\tau) = \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}.$$

J.T. Penedo, STP, arXiv:1806.11040

The modular forms of higher weight transform according to certain irreps of  $S_4$ . The dimension of the linear space of mod. forms of weight  $k$  is  $\dim \mathcal{M}_k(\Gamma(4)) = 2k + 1$ . At weight 4 there are 9 independent modular forms transforming in the 1, 2, 3 and 3' irreps of  $S_4$ :

$$\begin{aligned}
 Y_1^{(4)} &= Y_1 Y_2, & Y_2^{(4)} &= \begin{pmatrix} Y_2^2 \\ Y_1^2 \end{pmatrix}, \\
 Y_3^{(4)} &= \begin{pmatrix} Y_1 Y_4 - Y_2 Y_5 \\ Y_1 Y_5 - Y_2 Y_3 \\ Y_1 Y_3 - Y_2 Y_4 \end{pmatrix}, & Y_{3'}^{(4)} &= \begin{pmatrix} Y_1 Y_4 + Y_2 Y_5 \\ Y_1 Y_5 + Y_2 Y_3 \\ Y_1 Y_3 + Y_2 Y_4 \end{pmatrix}.
 \end{aligned}$$

J.T. Penedo, STP, arXiv:1806.11040

For the case of  $N = 4$  (i.e.,  $S_4$ ) we are going to consider further the weight 2 and the higher weight  $k \leq 10$  modular multiplets have been computed in the basis of  $S$  and  $T$  generators employed in arXiv:1806.11040. In this basis the triplet irreps of  $S$  and  $T$  to be used in our analysis read:

$$S = \pm \frac{1}{3} \begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \quad T = \pm \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$\omega = e^{i2\pi\tau/3}$ . The plus (minus) corresponds to the irrep 3 (3') of  $S_4$ .

In the employed basis we have:

$$ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

In certain cases of  $N = 3, 4, 5$  (i.e.,  $A_4, S_4, A_5$ ) it proves convenient to work in basis in which the generators  $S$  and  $T$  of these groups are represented by symmetric matrices,

$$\rho_{\mathbf{r}}(S) = \rho_{\mathbf{r}}^T(S), \quad \rho_{\mathbf{r}}(T) = \rho_{\mathbf{r}}^T(T),$$

for all irreducible representations  $\mathbf{r}$ .

The modular forms of levels  $N = 3, 4, 5$  and weights  $k \leq 10$  in the symmetric bases for  $S$  and  $T$  can be found in P.P. Novichkov et al., arXiv:1905.11970. We will be interested in the finite modular group  $\Gamma_4 \simeq S_4$ .

# Residual Symmetries

Residual symmetries arise whenever the VEV of the modulus  $\tau$  breaks  $\bar{\Gamma}$  only partially, i.e., the little group (stabiliser) of  $\langle\tau\rangle$  is non-trivial. There are only 3 inequivalent finite points with non-trivial little groups of  $\bar{\Gamma}$ :

$$\langle\tau\rangle = \mp 1/2 + i\sqrt{3}/2 \equiv \tau_{L,R}, \text{ and } \langle\tau\rangle = i \equiv \tau_C.$$

P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933.

Left cusp of the fundamental domain,

$$\langle\tau\rangle = -1/2 + i\sqrt{3}/2 \equiv \tau_L, \text{ is invariant under } ST: \tau = -1/(\tau + 1), Z_3^{ST} = \{I, ST, (ST)^2\} \text{ symmetry.}$$

The  $\langle\tau\rangle = i$  point is invariant under

$$S \text{ transformation } \tau = -1/\tau: Z_2^S = \{I, S\} \text{ symmetry.}$$

Right cusp of the fundamental domain,

$$\langle\tau\rangle = 1/2 + i\sqrt{3}/2 \equiv \tau_R = T \tau_L \text{ is invariant under } TS: \tau = 1 - 1/\tau, Z_3^{TS} = \{I, TS, (TS)^2\} \text{ symmetry.}$$

There is also infinite point  $\langle\tau\rangle = i\infty \equiv \tau_T$ , in which the subgroup  $Z_3^T = \{I, T, T^2\}$  of  $A_4$ ,  $Z_4^T = \{I, T, T^2, T^3\}$  of  $S_4$  ( $Z_N^T = \{I, T, T^2, T^3, \dots, T^{N-1}\}$  of  $\Gamma_N$ ) is preserved.

# The Framework

$\mathcal{N} = 1$  rigid (global) SUSY, the matter action  $\mathcal{S}$  reads:

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} K(\tau, \bar{\tau}, \psi, \bar{\psi}) + \left( \int d^4x d^2\theta W(\tau, \psi) + \text{h.c.} \right),$$

$K$  is the Kähler potential,  $W$  is the superpotential,  $\psi$  denotes a set of chiral supermultiplets  $\psi_i$ ,  $\theta$  and  $\bar{\theta}$  are Grassmann variables;

$\tau$  is the modulus chiral superfield, whose lowest component is the complex scalar field acquiring a VEV (we use in what follows the same notation  $\tau$  for the lowest complex scalar component of the modulus superfield and call this component also “modulus”).

$\tau$  and  $\psi_i$  transform under the action of  $\bar{\Gamma}$  in a certain way (S. Ferrara et al., PL B225 (1989) 363 and B233 (1989) 147). Assuming that  $\psi_i = \psi_i(x)$  transform in a certain irrep  $\mathbf{r}_i$  of  $\Gamma_N$ , the transformations read:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma} : \begin{cases} \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \\ \psi_i \rightarrow (c\tau + d)^{-k_i} \rho_{\mathbf{r}_i}(\gamma) \psi_i. \end{cases}$$

$\psi_i$  is not a multiplet of modular forms,  $(-k_i)$  can be odd and/or negative. Invariance of  $\mathcal{S}$  under these transformations implies (global SUSY):

$$\begin{cases} W(\tau, \psi) \rightarrow W(\tau, \psi), \\ K(\tau, \bar{\tau}, \psi, \bar{\psi}) \rightarrow K(\tau, \bar{\tau}, \psi, \bar{\psi}) + f_K(\tau, \psi) + \bar{f}_K(\bar{\tau}, \bar{\psi}). \end{cases}$$

The second line represents a Kähler transformation.

An example Kähler potential we will use in what follows, reads:

$$K(\tau, \bar{\tau}, \psi, \bar{\psi}) = -\Lambda_0^2 \log(-i\tau + i\bar{\tau}) + \sum_i \frac{|\psi_i|^2}{(-i\tau + i\bar{\tau})^{k_i}},$$

$\Lambda_0$  having mass dimension one. Under  $\Gamma_N$ ,  $(\tau - \bar{\tau}) \rightarrow (\tau - \bar{\tau})|c\tau + d|^{-2}$ , and

$$K(\tau, \bar{\tau}, \psi, \bar{\psi}) \rightarrow K(\tau, \bar{\tau}, \psi, \bar{\psi}) + \Lambda_0^2 (\log(c\tau + d) + \log(c\bar{\tau} + d))$$

Assuming only  $\tau$  acquires a VEV,  $K$  gives rise to kinetic terms for the scalar components of supermultiplets  $\tau$  and  $\psi$  of the form:

$$\frac{\Lambda_0^2}{(-i\tau + i\bar{\tau})^2} \partial_\mu \bar{\tau} \partial^\mu \tau + \sum_i \frac{\partial_\mu \bar{\psi}_i \partial^\mu \psi_i}{(-i\tau + i\bar{\tau})^{k_i}}.$$



The superpotential can be expanded in powers of  $\psi_i$ :

$$W(\tau, \psi) = \sum_n \sum_{\{i_1, \dots, i_n\}} \sum_s g_{i_1 \dots i_n, s} (Y_{i_1 \dots i_n, s}(\tau) \psi_{i_1} \dots \psi_{i_n})_{1, s} ,$$

$\mathbf{1}$  stands for an invariant singlet of  $\Gamma_N$ . For each set of  $n$  fields  $\{\psi_{i_1}, \dots, \psi_{i_n}\}$ , the index  $s$  labels the independent singlets. Each of these is accompanied by a coupling constant  $g_{i_1 \dots i_n, s}$  and is obtained using a modular multiplet  $Y_{i_1 \dots i_n, s}$  of the requisite weight. To ensure invariance of  $W$  under  $\Gamma_N$ ,  $Y_{i_1 \dots i_n, s}(\tau)$  must transform as:

$$Y(\tau) \xrightarrow{\gamma} (c\tau + d)^{k_Y} \rho_{\mathbf{r}_Y}(\gamma) Y(\tau) ,$$

$\mathbf{r}_Y$  is a representation of  $\Gamma_N$ , and  $k_Y$  and  $\mathbf{r}_Y$  are such that

$$k_Y = k_{i_1} + \dots + k_{i_n} , \quad (1)$$

$$\mathbf{r}_Y \otimes \mathbf{r}_{i_1} \otimes \dots \otimes \mathbf{r}_{i_n} \supset \mathbf{1} . \quad (2)$$

Thus,  $Y_{i_1 \dots i_n, s}(\tau)$  represents a multiplet of weight  $k_Y$  and level  $N$  modular forms transforming in the representation  $\mathbf{r}_Y$  of  $\Gamma_N$ .

# Lepton Flavour Models Based on $S_4$ (Seesaw Models without Flavons)

We assume that neutrino masses originate from the (supersymmetric) type I seesaw mechanism. The superpotential in the lepton sector reads

$$W = \alpha (E^c L H_d f_E(Y))_1 + g (N^c L H_u f_N(Y))_1 + \Lambda (N^c N^c f_M(Y))_1 ,$$

a sum over all independent invariant singlets with the coefficients  $\alpha = (\alpha, \alpha', \dots)$ ,  $g = (g, g', \dots)$  and  $\Lambda = (\Lambda, \Lambda', \dots)$  is implied.  $f_{E,N,M}(Y)$  denote the modular form multiplets required to ensure modular invariance.

For simplicity, we make the following assumptions:

- Higgs doublets  $H_u$  and  $H_d$  transform trivially under  $\Gamma_4$ ,  $\rho_u = \rho_d \sim 1$ , and  $k_u = k_d = 0$ ;
- lepton  $SU(2)$  doublets  $L_1, L_2, L_3$  furnish a 3-dim. irrep of  $\Gamma_4$ , i.e.,  $\rho_L \sim 3$  or  $3'$ ;
- neutral lepton gauge singlets  $N_1^c, N_2^c, N_3^c$  transform as a triplet of  $\Gamma_4$ ,  $\rho_N \sim 3$  or  $3'$ ;
- charged lepton  $SU(2)$  singlets  $E_1^c, E_2^c, E_3^c$  transform as singlets of  $\Gamma_4$ ,  $\rho_{1,2,3} \sim 1, 1'$ .

With these assumptions, we can rewrite the superpotential as

$$W = \sum_{i=1}^3 \alpha_i (E_i^c L f_{E_i}(Y))_1 H_d + g (N^c L f_N(Y))_1 H_u + \Lambda (N^c N^c f_M(Y))_1 ,$$

Assigning weights  $(-k_i)$ ,  $(-k_L)$ ,  $(-k_N)$  to  $E_i^c$ ,  $L$ ,  $N^c$ , and weights  $k_{\alpha_i}$ ,  $k_g$ ,  $k_\Lambda$  to the multiplets of modular forms  $f_{E_i}(Y)$ ,  $f_N(Y)$ ,  $f_M(Y)$ , modular invariance of the superpotential requires

$$\begin{cases} k_{\alpha_i} = k_i + k_L \\ k_g = k_N + k_L \\ k_\Lambda = 2k_N \end{cases} \Leftrightarrow \begin{cases} k_i = k_{\alpha_i} - k_g + k_\Lambda/2 \\ k_L = k_g - k_\Lambda/2 \\ k_N = k_\Lambda/2 \end{cases} .$$

By specifying the weights of the modular forms one obtains the weights of the matter superfields.

After modular symmetry breaking, the matrices of charged lepton and neutrino Yukawa couplings,  $\lambda$  and  $\mathcal{Y}$ , as well as the Majorana mass matrix  $M$  for heavy neutrinos, are generated:

$$W = \lambda_{ij} E_i^c L_j H_d + \mathcal{Y}_{ij} N_i^c L_j H_u + \frac{1}{2} M_{ij} N_i^c N_j^c ,$$

a sum over  $i, j = 1, 2, 3$  is assumed. After integrating out  $N^c$  and after EWS breaking, the charged lepton mass matrix  $M_e$  and the light neutrino Majorana mass matrix  $M_\nu$  are generated (we work in the L-R convention for the charged lepton mass term and the R-L convention for the light and heavy neutrino Majorana mass terms):

$$\begin{aligned} M_e &= v_d \lambda^\dagger, & v_d &\equiv H_d^0, \\ M_\nu &= -v_u^2 \mathcal{Y}^T M^{-1} \mathcal{Y}, & v_u &\equiv H_u^0. \end{aligned}$$

# The Majorana mass term for heavy neutrinos

Assume  $k_\Lambda = 0$ , i.e., no non-trivial modular forms are present in  $\Lambda(N^c N^c f_M(Y))_1$ ,  $k_N = 0$ , and for both  $\rho_N \sim 3$  or  $\rho_N \sim 3'$

$$(N^c N^c)_1 = N_1^c N_1^c + N_2^c N_3^c + N_3^c N_2^c,$$

leading to the following mass matrix for heavy neutrinos:

$$M = 2\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{for } k_\Lambda = 0.$$

The spectrum of heavy neutrino masses is degenerate; the only free parameter is the overall scale  $\Lambda$ , which can be rendered real. The Majorana mass term conserves a “non-standard” lepton charge and two of the three heavy Majorana neutrinos with definite mass form a Dirac pair.

C.N. Leung, STP, 1983

# The neutrino Yukawa couplings

The lowest non-trivial weight,  $k_g = 2$ , leads to

$$g (N^c L Y_2)_1 H_u + g' (N^c L Y_{3'})_1 H_u.$$

There are 4 possible assignments of  $\rho_N$  and  $\rho_L$  we consider. Two of them, namely  $\rho_N = \rho_L \sim \mathbf{3}$  and  $\rho_N = \rho_L \sim \mathbf{3}'$  give the following form of  $\mathcal{Y}$ :

$$\mathcal{Y} = g \left[ \begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & Y_2 & 0 \\ Y_2 & 0 & Y_1 \end{pmatrix} + \frac{g'}{g} \begin{pmatrix} 0 & Y_5 & -Y_4 \\ -Y_5 & 0 & Y_3 \\ Y_4 & -Y_3 & 0 \end{pmatrix} \right], \quad \text{for } k_g = 2 \quad \text{and} \quad \rho_N = \rho_L.$$

The two remaining combinations,  $(\rho_N, \rho_L) \sim (\mathbf{3}, \mathbf{3}')$  and  $(\mathbf{3}', \mathbf{3})$ , lead to:

$$\mathcal{Y} = g \left[ \begin{pmatrix} 0 & -Y_1 & Y_2 \\ -Y_1 & Y_2 & 0 \\ Y_2 & 0 & -Y_1 \end{pmatrix} + \frac{g'}{g} \begin{pmatrix} 2Y_3 & -Y_5 & -Y_4 \\ -Y_5 & 2Y_4 & -Y_3 \\ -Y_4 & -Y_3 & 2Y_5 \end{pmatrix} \right], \quad \text{for } k_g = 2 \quad \text{and} \quad \rho_N \neq \rho_L.$$

In both cases, up to an overall factor, the matrix  $\mathcal{Y}$  depends on one complex parameter  $g'/g$  and the VEV  $\tau$ .

# The charged lepton Yukawa couplings

Since we consider  $\rho_i \sim 1$  or  $1'$  and  $\rho_L \sim 3$  or  $3'$ , we have four possible combinations  $\rho_i \otimes \rho_L$ . None of them contain the invariant singlet. Thus, the weights  $k_{\alpha_i}$  cannot be zero, i.e., they are strictly positive,  $k_{\alpha_i} > 0$ . Moreover,  $f_{E_i}(Y)$  should transform in  $3$  if  $(\rho_i, \rho_L) \sim (1, 3)$  or  $(1', 3')$ , and in  $3'$  if  $(\rho_i, \rho_L) \sim (1, 3')$  or  $(1', 3)$ . Thus, for each  $i = 1, 2, 3$ , we have

$$\alpha_i (E_i^c L f_{E_i}(Y))_1 H_d = E_i^c \sum_a \alpha_{i,a} \left[ L_1 \left( Y_a^{(k_{\alpha_i})} \right)_1 + L_2 \left( Y_a^{(k_{\alpha_i})} \right)_3 + L_3 \left( Y_a^{(k_{\alpha_i})} \right)_2 \right] H_d,$$

where  $Y_a^{(k_{\alpha_i})}$  are independent triplets ( $3$  or  $3'$  depending on  $\rho_i$  and  $\rho_L$ ) of weight  $k_{\alpha_i}$ .  $k_{\alpha_i} = 2$ ,  $i = 1, 2, 3$  or  $i = 1, 2$  is not phenomenologically viable (leads to two or one zero mass charged leptons). The minimal (in terms of weights) viable possibility is defined by  $k_{\alpha_i} = 2$  and  $k_{\alpha_j} = k_{\alpha_p} = 4$ , for  $j \neq p$ , with  $\rho_j \neq \rho_p$ . Possible since there are two triplets of weight 4,  $Y_3^{(4)}$  and  $Y_{3'}^{(4)}$ .

Then the relevant part of  $W$ ,  $W_e$ , can take 6 different forms which lead to the same matrix  $U_e$  diagonalising  $M_e M_e^\dagger = v_d^2 \lambda^\dagger \lambda$ , and thus do not lead to new results for the PMNS matrix. We give just one of these 6 forms corresponding to  $\rho_L = 3$ ,  $\rho_1 = 1'$ ,  $\rho_2 = 1$ ,  $\rho_3 = 1'$ :

$$\alpha (E_1^c L Y_{3'})_1 H_d + \beta \left( E_2^c L Y_3^{(4)} \right)_1 H_d + \gamma \left( E_3^c L Y_{3'}^{(4)} \right)_1 H_d.$$

**This leads leads to**

$$\lambda = \begin{pmatrix} \alpha Y_3 & \alpha Y_5 & \alpha Y_4 \\ \beta (Y_1 Y_4 - Y_2 Y_5) & \beta (Y_1 Y_3 - Y_2 Y_4) & \beta (Y_1 Y_5 - Y_2 Y_3) \\ \gamma (Y_1 Y_4 + Y_2 Y_5) & \gamma (Y_1 Y_3 + Y_2 Y_4) & \gamma (Y_1 Y_5 + Y_2 Y_3) \end{pmatrix},$$

**In this “minimal” example the matrix  $\lambda$  depends on 3 free parameters,  $\alpha$ ,  $\beta$  and  $\gamma$ , which can be rendered real by re-phasing of the charged lepton fields, and the complex  $\tau$ .**

**We recall that**

$$M_e = v_d \lambda^\dagger, \quad v_d \equiv H_d^0, \\ M_\nu = -v_u^2 \mathcal{Y}^T M^{-1} \mathcal{Y}, \quad v_u \equiv H_u^0.$$



Parameters of the model:  $\alpha, \beta, \gamma, g^2/\Lambda$  – real;  $g'$  and VEV of  $\tau$  – complex, i.e., 6 real parameters + 2 phases for description of 12 observables (3 charged lepton masses, 3 neutrino masses, 3 mixing angles and 3 CPV phases). Excellent description of the data is obtained also for real  $g'$  (i.e., 6 real parameters + 1 phase).

The 3 real parameters  $v_d\alpha, \beta/\alpha, \gamma/\alpha$  – fixed by fitting  $m_e, m_\mu$  and  $m_\tau$ . The remaining 3 real parameters and 2 (1) phases –  $v_u^2 g^2/\Lambda, |g'/g|, |\tau|$  and  $\arg(g'/g), \arg \tau$  ( $\arg \tau$ ) – describe the 9  $\nu$  observables, **3  $\nu$  masses, 3 mixing angles and 3 CPV phases.**

The model considered leads to testable predictions for  $\min(m_j)$  ( $\sum_i m_i$ ), type of the  $\nu$  mass spectrum (NO or IO), the CPV Dirac and Majorana phases,  $|\langle m \rangle|, \theta_{23}$ , as well as of correlations between different observables.

## Numerical Analysis

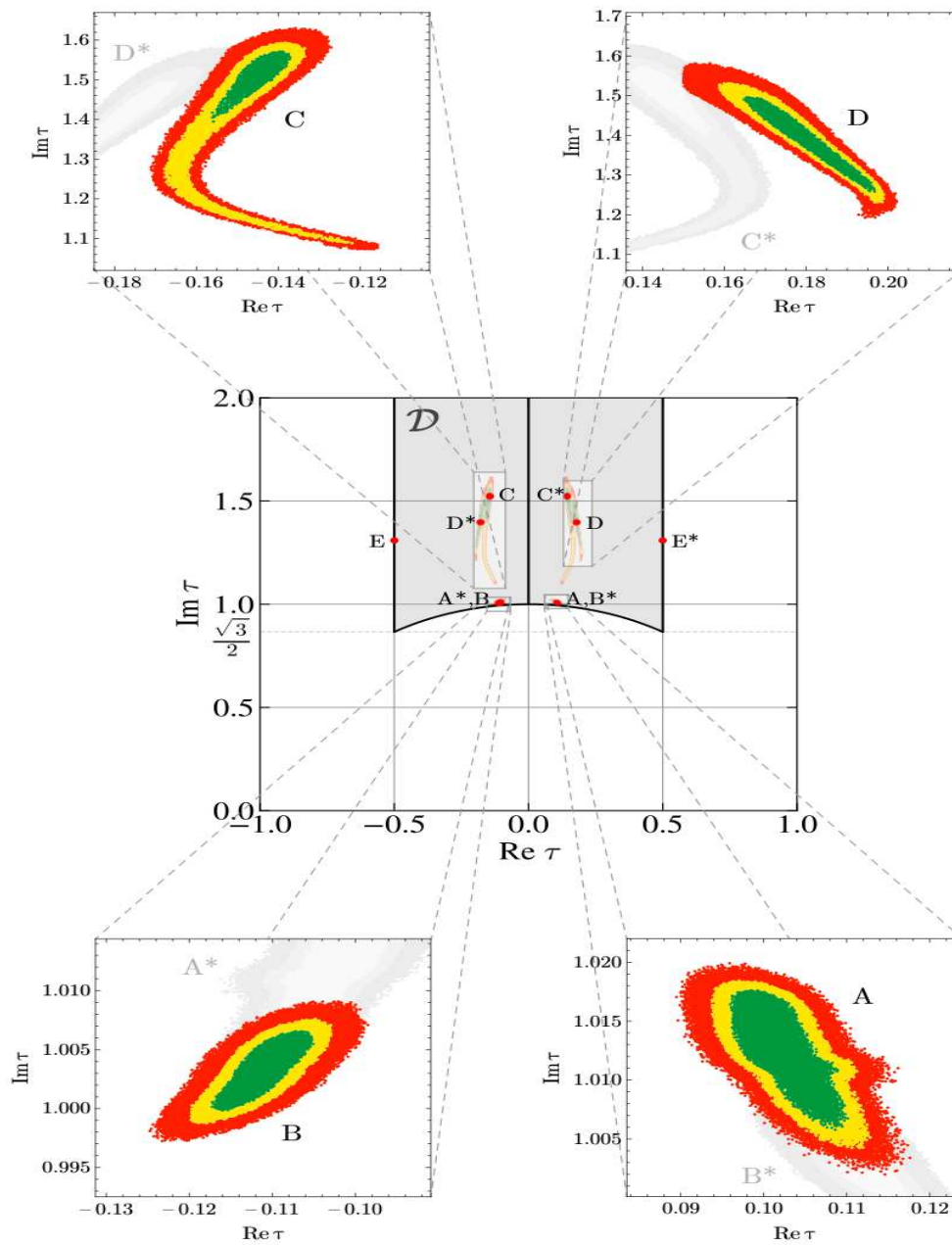
Each model depends on a set of dimensionless parameters

$$p_i = (\tau, \beta/\alpha, \gamma/\alpha, g'/g, \dots, \Lambda'/\Lambda, \dots),$$

which determine dimensionless observables (mass ratios, mixing angles and phases), and two overall mass scales:  $v_d \alpha$  for  $M_e$  and  $v_u^2 g^2 / \Lambda$  for  $M_\nu$ . Phenomenologically viable models are those that lead to values of observables which are in close agreement with the experimental results summarised in the Table below. We assume also to be in a regime in which the running of neutrino parameters is negligible.

Observable	Best fit value and $1\sigma$ range	
$m_e/m_\mu$	$0.0048 \pm 0.0002$	
$m_\mu/m_\tau$	$0.0565 \pm 0.0045$	
	NO	IO
$\delta m^2/(10^{-5} \text{ eV}^2)$	$7.34^{+0.17}_{-0.14}$	
$ \Delta m^2 /(10^{-3} \text{ eV}^2)$	$2.455^{+0.035}_{-0.032}$	$2.441^{+0.033}_{-0.035}$
$r \equiv \delta m^2/ \Delta m^2 $	$0.0299 \pm 0.0008$	$0.0301 \pm 0.0008$
$\sin^2 \theta_{12}$	$0.304^{+0.014}_{-0.013}$	$0.303^{+0.014}_{-0.013}$
$\sin^2 \theta_{13}$	$0.0214^{+0.0009}_{-0.0007}$	$0.0218^{+0.0008}_{-0.0007}$
$\sin^2 \theta_{23}$	$0.551^{+0.019}_{-0.070}$	$0.557^{+0.017}_{-0.024}$
$\delta/\pi$	$1.32^{+0.23}_{-0.18}$	$1.52^{+0.14}_{-0.15}$

**Best fit values and  $1\sigma$  ranges for neutrino oscillation parameters, obtained in the global analysis of F. Capozzi et al., arXiv:1804.09678, and for charged-lepton mass ratios, given at the scale  $2 \times 10^{16}$  GeV with the  $\tan\beta$  averaging described in F. Feruglio, arXiv:1706.08749 obtained from G.G. Ross and M. Serna, arXiv:0704.1248. The parameters entering the definition of  $r$  are  $\delta m^2 \equiv m_2^2 - m_1^2$  and  $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ . The best fit value and  $1\sigma$  range of  $\delta$  did not drive the numerical searches here reported.**



P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933

	Best fit value	$2\sigma$ range	$3\sigma$ range
$\text{Re } \tau$	$\pm 0.1045$	$\pm(0.09597 - 0.1101)$	$\pm(0.09378 - 0.1128)$
$\text{Im } \tau$	1.01	1.006 – 1.018	1.004 – 1.018
$\beta/\alpha$	9.465	8.247 – 11.14	7.693 – 12.39
$\gamma/\alpha$	0.002205	0.002032 – 0.002382	0.001941 – 0.002472
$\text{Re } g'/g$	0.233	-0.02383 – 0.387	-0.02544 – 0.4417
$\text{Im } g'/g$	$\pm 0.4924$	$\pm(-0.592 - 0.5587)$	$\pm(-0.6046 - 0.5751)$
$v_d \alpha$ [MeV]	53.19		
$v_u^2 g^2/\Lambda$ [eV]	0.00933		
$m_e/m_\mu$	0.004802	0.004418 – 0.005178	0.00422 – 0.005383
$m_\mu/m_\tau$	0.0565	0.048 – 0.06494	0.04317 – 0.06961
$r$	0.02989	0.02836 – 0.03148	0.02759 – 0.03224
$\delta m^2$ [ $10^{-5}$ eV <sup>2</sup> ]	7.339	7.074 – 7.596	6.935 – 7.712
$ \Delta m^2 $ [ $10^{-3}$ eV <sup>2</sup> ]	2.455	2.413 – 2.494	2.392 – 2.513
$\sin^2 \theta_{12}$	0.305	0.2795 – 0.3313	0.2656 – 0.3449
$\sin^2 \theta_{13}$	0.02125	0.01988 – 0.02298	0.01912 – 0.02383
$\sin^2 \theta_{23}$	0.551	0.4846 – 0.5846	0.4838 – 0.5999
Ordering	NO		
$m_1$ [eV]	0.01746	0.01196 – 0.02045	0.01185 – 0.02143
$m_2$ [eV]	0.01945	0.01477 – 0.02216	0.01473 – 0.02307
$m_3$ [eV]	0.05288	0.05099 – 0.05405	0.05075 – 0.05452
$\sum_i m_i$ [eV]	0.0898	0.07774 – 0.09661	0.07735 – 0.09887
$ \langle m \rangle $ [eV]	0.01699	0.01188 – 0.01917	0.01177 – 0.02002
$\delta/\pi$	$\pm 1.314$	$\pm(1.266 - 1.95)$	$\pm(1.249 - 1.961)$
$\alpha_{21}/\pi$	$\pm 0.302$	$\pm(0.2821 - 0.3612)$	$\pm(0.2748 - 0.3708)$
$\alpha_{31}/\pi$	$\pm 0.8716$	$\pm(0.8162 - 1.617)$	$\pm(0.7973 - 1.635)$
$N\sigma$	0.02005		

Best fit values along with  $2\sigma$  and  $3\sigma$  ranges of the parameters and observables in cases A and A\*, (which refer to  $(k_\Lambda, k_g) = (0, 2)$  and  $\tau = \pm 0.1045 + i 1.01$ ).

	Best fit value	$2\sigma$ range	$3\sigma$ range
$\text{Re } \tau$	$\mp 0.109$	$\mp(0.1051 - 0.1172)$	$\mp(0.103 - 0.1197)$
$\text{Im } \tau$	1.005	0.9998 – 1.007	0.9988 – 1.008
$\beta/\alpha$	0.03306	0.02799 – 0.03811	0.02529 – 0.04074
$\gamma/\alpha$	0.0001307	0.0001091 – 0.0001538	0.0000982 – 0.0001663
$\text{Re } g'/g$	0.4097	0.3513 – 0.5714	0.3241 – 0.5989
$\text{Im } g'/g$	$\mp 0.5745$	$\mp(0.5557 - 0.5932)$	$\mp(0.5436 - 0.5944)$
$v_d \alpha$ [MeV]	893.2		
$v_u^2 g^2/\Lambda$ [eV]	0.008028		
$m_e/m_\mu$	0.004802	0.004425 – 0.005175	0.004211 – 0.005384
$m_\mu/m_\tau$	0.05649	0.04785 – 0.06506	0.04318 – 0.06962
$r$	0.0299	0.02838 – 0.03144	0.02757 – 0.03223
$\delta m^2$ [ $10^{-5}$ eV <sup>2</sup> ]	7.34	7.078 – 7.59	6.932 – 7.71
$ \Delta m^2 $ [ $10^{-3}$ eV <sup>2</sup> ]	2.455	2.414 – 2.494	2.393 – 2.514
$\sin^2 \theta_{12}$	0.305	0.2795 – 0.3314	0.2662 – 0.3455
$\sin^2 \theta_{13}$	0.02125	0.0199 – 0.02302	0.01914 – 0.02383
$\sin^2 \theta_{23}$	0.551	0.4503 – 0.5852	0.4322 – 0.601
Ordering	NO		
$m_1$ [eV]	0.02074	0.01969 – 0.02374	0.01918 – 0.02428
$m_2$ [eV]	0.02244	0.02148 – 0.02522	0.02101 – 0.02574
$m_3$ [eV]	0.05406	0.05345 – 0.05541	0.05314 – 0.05577
$\sum_i m_i$ [eV]	0.09724	0.09473 – 0.1043	0.0935 – 0.1056
$ \langle m \rangle $ [eV]	0.01983	0.01889 – 0.02229	0.01847 – 0.02275
$\delta/\pi$	$\pm 1.919$	$\pm(1.895 - 1.968)$	$\pm(1.882 - 1.977)$
$\alpha_{21}/\pi$	$\pm 1.704$	$\pm(1.689 - 1.716)$	$\pm(1.681 - 1.722)$
$\alpha_{31}/\pi$	$\pm 1.539$	$\pm(1.502 - 1.605)$	$\pm(1.484 - 1.618)$
$N\sigma$	0.02435		

Best fit values along with  $2\sigma$  and  $3\sigma$  ranges of the parameters and observables in cases B and B\*, (which refer to  $(k_\Lambda, k_g) = (0, 2)$  and  $\tau = \pm 0.109 + i 1.005$ ).

	Best fit value	$2\sigma$ range	$3\sigma$ range
$\text{Re } \tau$	$\mp 0.1435$	$\mp(0.137 - 0.1615)$	$\mp(0.1222 - 0.168)$
$\text{Im } \tau$	1.523	1.147 – 1.572	1.088 – 1.594
$\beta/\alpha$	17.82	10.99 – 21.38	9.32 – 23.66
$\gamma/\alpha$	0.003243	0.002518 – 0.003565	0.00227 – 0.003733
$\text{Re } g'/g$	-0.8714	-(0.8209 – 1.132)	-(0.7956 – 1.148)
$\text{Im } g'/g$	$\mp 2.094$	$\mp(1.439 - 2.157)$	$\mp(1.409 - 2.182)$
$v_d \alpha$ [MeV]	71.26		
$v_u^2 g^2/\Lambda$ [eV]	0.008173		
$m_e/m_\mu$	0.004797	0.00442 – 0.005183	0.004215 – 0.005378
$m_\mu/m_\tau$	0.05655	0.04806 – 0.06507	0.04348 – 0.0698
$r$	0.0301	0.02857 – 0.03162	0.0278 – 0.03246
$\delta m^2$ [ $10^{-5}$ eV $^2$ ]	7.346	7.084 – 7.589	6.946 – 7.717
$ \Delta m^2 $ [ $10^{-3}$ eV $^2$ ]	2.44	2.4 – 2.479	2.377 – 2.498
$\sin^2 \theta_{12}$	0.303	0.278 – 0.3288	0.2657 – 0.3436
$\sin^2 \theta_{13}$	0.02175	0.02035 – 0.0234	0.01957 – 0.0242
$\sin^2 \theta_{23}$	0.5571	0.4905 – 0.588	0.4551 – 0.6026
<b>Ordering</b>	<b>IO</b>		
$m_1$ [eV]	0.0513	0.04938 – 0.0518	0.04882 – 0.05207
$m_2$ [eV]	0.05201	0.05012 – 0.05248	0.04958 – 0.05274
$m_3$ [eV]	0.01512	0.00576 – 0.01594	0.00316 – 0.0163
$\sum_i m_i$ [eV]	0.1184	0.1053 – 0.1201	0.102 – 0.1208
$ \langle m \rangle $ [eV]	0.0263	0.0239 – 0.04266	0.02288 – 0.04551
$\delta/\pi$	$\pm 1.098$	$\pm(1.026 - 1.278)$	$\pm(0.98 - 1.289)$
$\alpha_{21}/\pi$	$\pm 1.241$	$\pm(1.162 - 1.651)$	$\pm(1.113 - 1.758)$
$\alpha_{31}/\pi$	$\pm 0.2487$	$\pm(0.1474 - 0.3168)$	$\pm(0.069 - 0.346)$
$N\sigma$	0.0357		

Best fit values along with  $2\sigma$  and  $3\sigma$  ranges of the parameters and observables in cases C and C\*, (which refer to  $(k_\Lambda, k_g) = (0, 2)$  and  $\tau = \pm 0.1453 + i 1.523$ ).

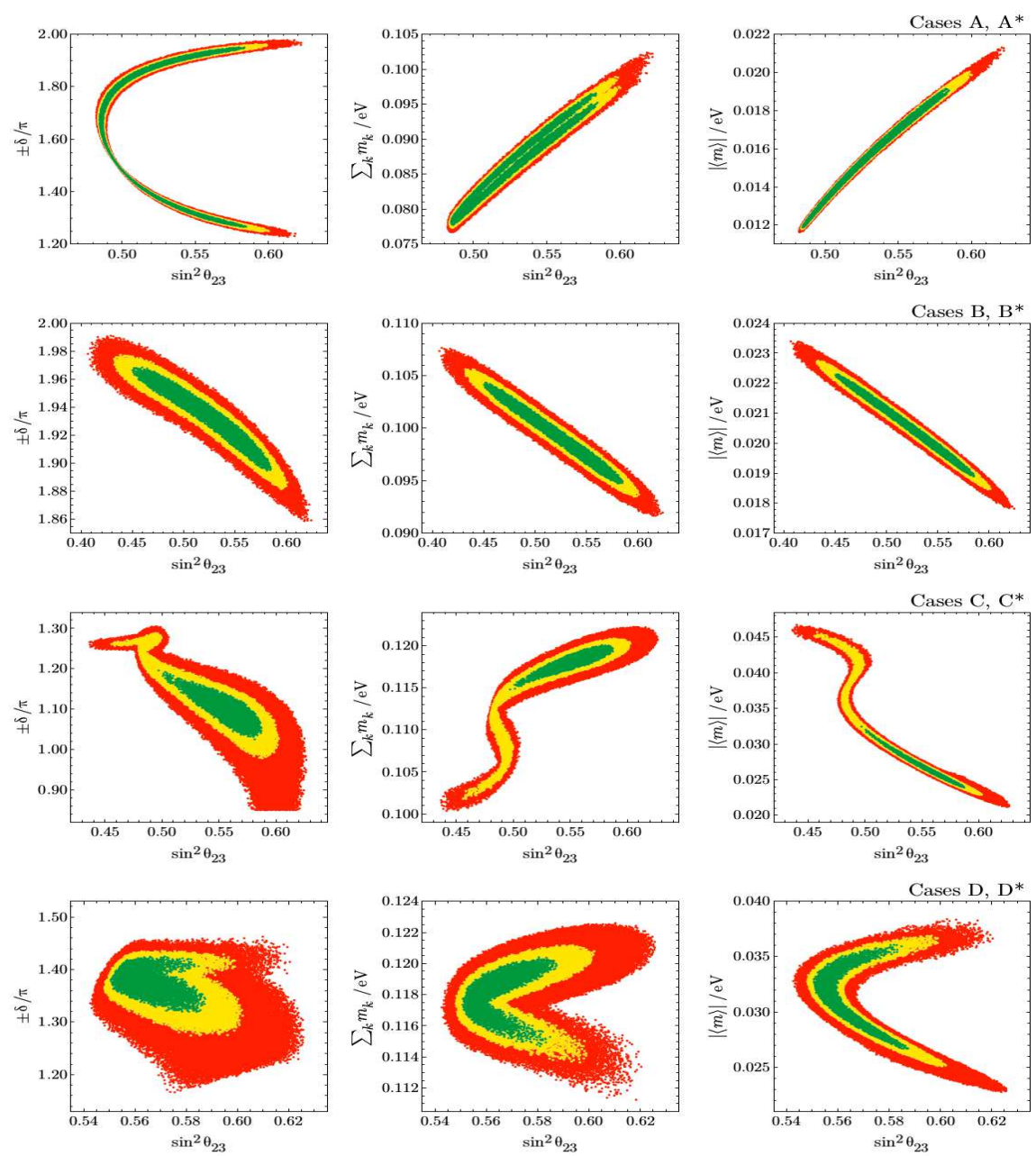
	Best fit value	$2\sigma$ range	$3\sigma$ range
$\text{Re } \tau$	$\pm 0.179$	$\pm(0.165 - 0.1963)$	$\pm(0.1589 - 0.199)$
$\text{Im } \tau$	1.397	1.262 – 1.496	1.236 – 1.529
$\beta/\alpha$	15.35	11.67 – 18.66	10.79 – 21.09
$\gamma/\alpha$	0.002924	0.002582 – 0.003289	0.002443 – 0.003459
$\text{Re } g'/g$	-1.32	-(1.189 – 1.438)	-(1.131 – 1.447)
$\text{Im } g'/g$	$\pm 1.733$	$\pm(1.357 - 1.948)$	$\pm(1.306 - 2.017)$
$v_d \alpha$ [MeV]	68.42		
$v_u^2 g^2/\Lambda$ [eV]	0.00893		
$m_e/m_\mu$	0.004786	0.004431 – 0.005186	0.004221 – 0.005386
$m_\mu/m_\tau$	0.0554	0.0481 – 0.06502	0.04343 – 0.06968
$r$	0.03023	0.02859 – 0.03163	0.02775 – 0.03244
$\delta m^2$ [ $10^{-5}$ eV <sup>2</sup> ]	7.367	7.088 – 7.59	6.937 – 7.713
$ \Delta m^2 $ [ $10^{-3}$ eV <sup>2</sup> ]	2.437	2.4 – 2.479	2.378 – 2.499
$\sin^2 \theta_{12}$	0.3031	0.2791 – 0.3286	0.2657 – 0.3436
$\sin^2 \theta_{13}$	0.02184	0.02038 – 0.02337	0.01954 – 0.0242
$\sin^2 \theta_{23}$	0.5577	0.5509 – 0.5869	0.5482 – 0.6013
<b>Ordering</b>	<b>IO</b>		
$m_1$ [eV]	0.05122	0.05051 – 0.05185	0.05023 – 0.05212
$m_2$ [eV]	0.05193	0.05125 – 0.05253	0.05098 – 0.05279
$m_3$ [eV]	0.01495	0.01293 – 0.01613	0.01223 – 0.01649
$\sum_i m_i$ [eV]	0.1181	0.1149 – 0.1203	0.1139 – 0.1212
$ \langle m \rangle $ [eV]	0.03104	0.02666 – 0.03597	0.02515 – 0.03677
$\delta/\pi$	$\pm 1.384$	$\pm(1.32 - 1.4245)$	$\pm(1.271 - 1.437)$
$\alpha_{21}/\pi$	$\pm 1.343$	$\pm(1.227 - 1.457)$	$\pm(1.171 - 1.479)$
$\alpha_{31}/\pi$	$\pm 0.806$	$\pm(0.561 - 1.092)$	$\pm(0.448 - 1.149)$
$N\sigma$	0.3811		

Best fit values along with  $2\sigma$  and  $3\sigma$  ranges of the parameters and observables in cases D and D\*, (which refer to  $(k_\Lambda, k_g) = (0, 2)$  and  $\tau = \pm 0.179 + i 1.397$ ).

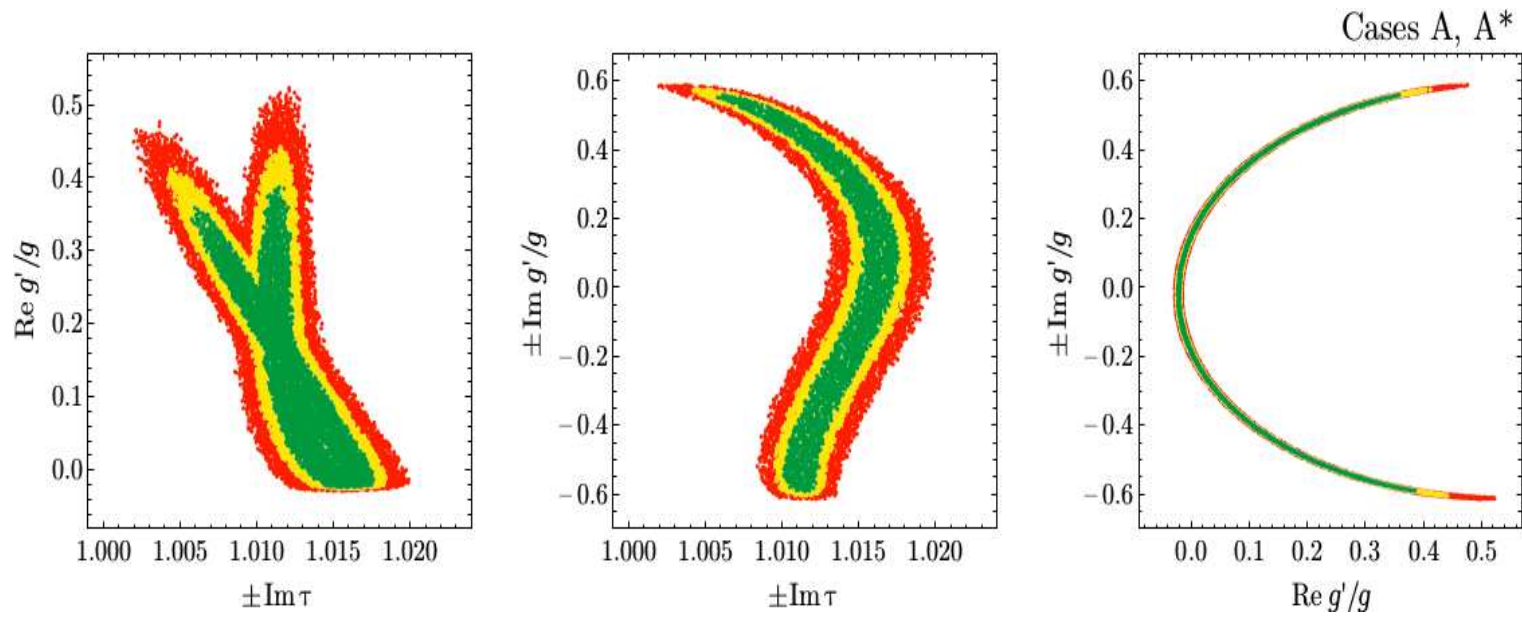


	Best fit value	$3\sigma$ range
$\text{Re } \tau$	$\mp 0.4996$	$\mp(0.48 - 0.5084)$
$\text{Im } \tau$	1.309	1.246 – 1.385
$\beta/\alpha$	0.000243	0.0002004 – 0.0002864
$\gamma/\alpha$	0.03335	0.02799 – 0.03926
$\text{Re } g'/g$	-0.06454	-(0.01697 – 0.1215)
$\text{Im } g'/g$	$\mp 0.569$	$\mp(0.4572 - 0.6564)$
$v_d \alpha$ [MeV]	1125	
$v_u^2 g^2/\Lambda$ [eV]	0.0174	
$m_e/m_\mu$	0.004797	0.004393 – 0.005197
$m_\mu/m_\tau$	0.05626	0.04741 – 0.0654
$r$	0.02985	0.02826 – 0.03146
$\delta m^2$ [ $10^{-5}$ eV <sup>2</sup> ]	7.332	7.055 – 7.593
$ \Delta m^2 $ [ $10^{-3}$ eV <sup>2</sup> ]	2.456	2.413 – 2.497
$\sin^2 \theta_{12}$	0.311	0.2895 – 0.3375
$\sin^2 \theta_{13}$	0.02185	0.02041 – 0.02351
$\sin^2 \theta_{23}$	0.4469	0.43 – 0.4614
Ordering	NO	
$m_1$ [eV]	0.01774	0.01703 – 0.01837
$m_2$ [eV]	0.0197	0.01906 – 0.02025
$m_3$ [eV]	0.05299	0.05251 – 0.05346
$\sum_i m_i$ [eV]	0.09043	0.08874 – 0.09195
$ \langle m \rangle $ [eV]	0.006967	0.006482 – 0.007288
$\delta/\pi$	$\pm 1.601$	$\pm(1.287 - 1.828)$
$\alpha_{21}/\pi$	$\pm 1.093$	$\pm(0.8593 - 1.178)$
$\alpha_{31}/\pi$	$\pm 0.7363$	$\pm(0.3334 - 0.9643)$
$N\sigma$	2.147	

Best fit values along with  $2\sigma$  and  $3\sigma$  ranges of the parameters and observables in cases E and E\*, (which refer to  $(k_\Lambda, k_g) = (0, 2)$  and  $\tau = \pm 0.4996 + i 1.309$ ).



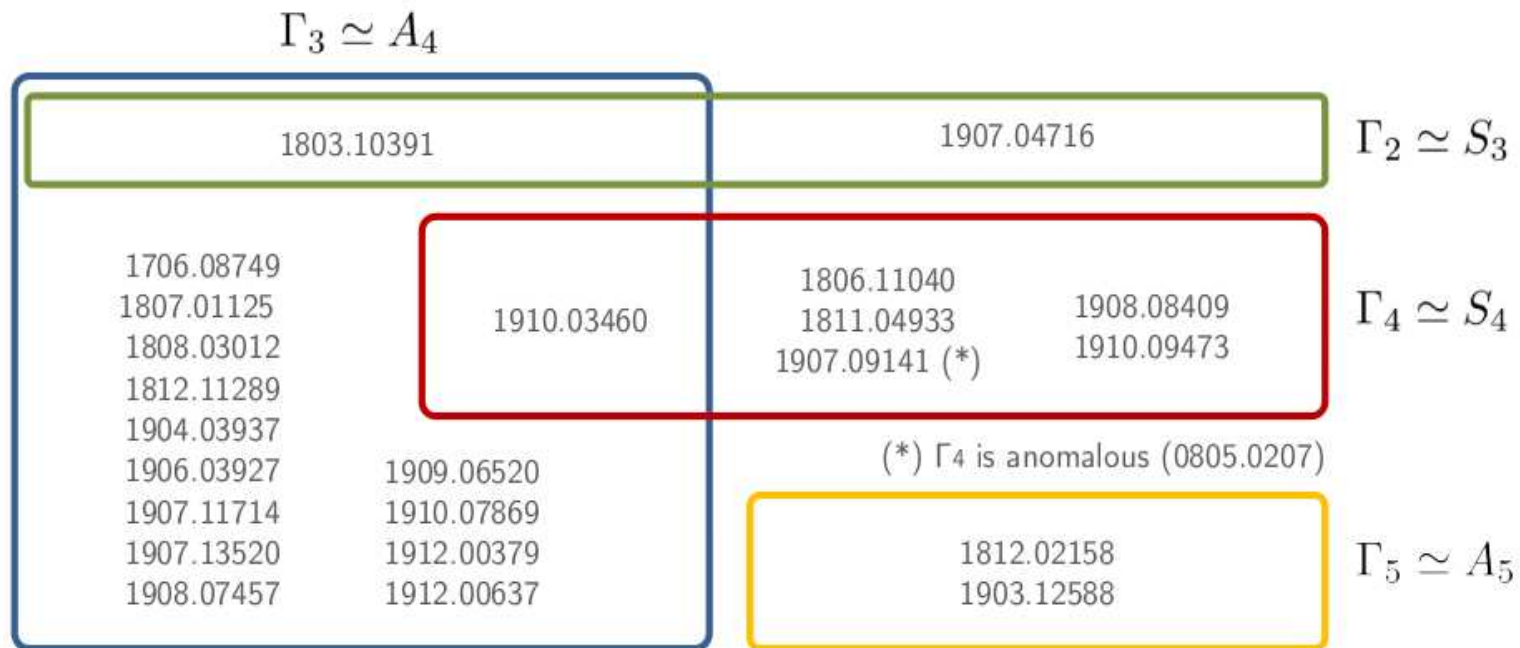
P.P. Novichkov et al., arXiv:1811.04933



P.P. Novichkov et al., arXiv:1811.04933

# Literature overview

## Bottom-up approach in the **lepton sector**



40 papers have appeared since the seminal 1706.08749  
(as of 11 December 2019)

J. Penedo, talk at SISSA, 11/12/2019

# Literature overview

## Bottom-up approach in the **lepton sector** (cont.)

- Models with and without flavons
- Neutrino mass origin: Weinberg, Seesaw (I, II, inverse), Dirac, 1 or 2 loops
- Dark matter, leptogenesis, texture zeros, 2RHN, ...

## Bottom-up approach for **quarks** and **quarks+leptons**

$$\left. \begin{array}{l} \Gamma_2 \simeq S_3 \\ \Gamma_3 \simeq A_4 \end{array} \right\} \begin{array}{l} 1812.09677 - \text{just quarks, with Higgs triplets} \\ 1812.11072 - \text{no unification, a modular symmetry for each sector} \\ 1905.13421 - \text{both sectors fit with one tau} \\ 1906.10341 - \text{SU(5) GUT with a very large number of parameters} \end{array}$$

J. Penedo, talk at SISSA, 11/12/2019

# Literature overview

## Generalisations of the bottom-up approach

- Mass hierarchies using flavons and weights as FN charges [1908.11867]
- Multiple modular symmetries [1906.02208, 1908.02770]
- Tentative studies of modulus stabilisation [1909.05139, 1910.11553]
- Odd weight modular forms:  $\text{PSL}(2, \mathbb{Z}) \rightarrow \text{SL}(2, \mathbb{Z})$  [1907.01488]
- Addition of a generalised CP symmetry [1905.11970, 1910.11553]

J. Penedo, talk at SISSA, 11/12/2019

**M. Tanimoto, T. Kobayashi et al.:**

19120.11553, 1909.05139, 1907.09141, 1906.10341, 1812.11072,  
1808.03012

**M. Tanimoto et al.:** 1812.09677, 1812.11289

**T. Kobayashi et al.:**

2001.07972, 19120.00637, 1911.01930, 1904.07546, 1811.11384,  
1804.06644, 1803.10391

## Conclusions.

- Understanding the origin of the pattern of neutrino mixing and of neutrino mass squared differences that emerged from the neutrino oscillation data in the recent years is one of the most challenging problems in neutrino physics. It is part of the fundamental problem of understanding the origin of flavour in particle physics.
- The modular invariance (finite modular group symmetries) is a new elegant and very promising approach to the flavour problem. It has been successfully applied to the lepton flavour problem.
- In its minimal version the approach involves just one complex scalar field – the modulus  $\tau$ , and a certain rather small number of constant parameters. The modular symmetry is broken by the the VEV of  $\tau$ .
- The models of lepton flavour based of finite modular symmetries, lead to testable predictions for  $\min(m_j)$ , type of the neutrino mass spectrum (NO or IO),  $\sum_i m_i$ , the CPV Dirac and Majorana phases,  $|\langle m \rangle|$ ,  $\theta_{23}$ , as well as of correlations between different observables.
- The modular invariance approach to the flavour problem is still at the early stage of its development, with many aspects still to be understood.