NTNU
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Pion condensation - χpt versus lattice

XQCD 2019, Tokyo June 26

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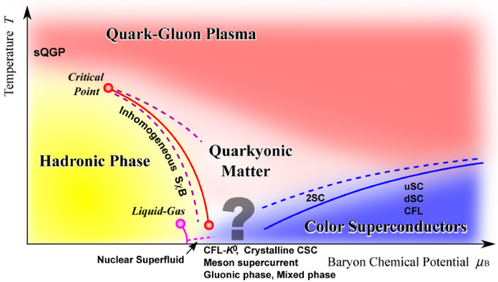
Reference: arXiv:1904.03887

Introduction

— **QCD phase diagram**



— **Only a few exact results**

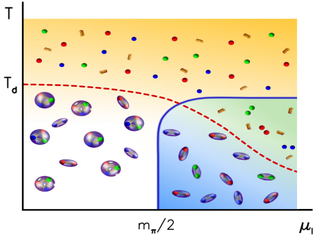
- Quark-gluon plasma at asymptotically high temperature
- Color-flavor locked phase at asymptotically large baryon density.

— **Sign problem at finite baryon chemical potential (talks by Nishimura, Fujii, Scherzer...)**

— **No sign problem at finite magnetic field B or finite isospin μ_I with $\mu_B = 0$**

$$\begin{aligned}
Z &= \int \mathcal{D}A_\mu e^{-S_g \det(\not{D}+m_f)} \, , \\
\det\left(\not{D} + m_q - \mu_q \gamma^0\right) &= \det\left[(iX + \mu_q)(iX^\dagger + \mu_q) + m_q^2\right] \, , \\
\mu_I &= \frac{1}{2}(\mu_u - \mu_d) \, , \\
\mu_B &= \frac{3}{2}(\mu_u + \mu_d) \, .
\end{aligned}$$

— **This talk** [2]



- Properties of the BEC phase at $T = 0$ using χpt (see also poster by Schmalzbauer)

[2] B. B. Brandt, G. Endrodi, and S. Schmalzbauer, Phys.Rev. D **97**, 054514 (2018).

χpt at finite isospin μ_I

- **Low-energy effective theory for QCD based on symmetries and relevant degrees of freedom**

- **Two-flavor QCD, pions and $SU(2)_L \times SU(2)_R$**

- **Leading order Lagrangian and addition of quark chemical potential**

$$
\begin{aligned}
\mathcal{L}_2 &= \frac{f^2}{4}\,\mathrm{Tr}\left[\nabla^\mu \Sigma^\dagger \nabla_\mu \Sigma\right] + \frac{f^2 m^2}{4}\,\mathrm{Tr}\left[\Sigma + \Sigma^\dagger\right] \,, \\
\nabla_\mu \Sigma &\equiv \partial_\mu \Sigma - i\,[v_\mu, \Sigma] \,, \\
v_\mu &= \delta_{\mu,0}\mathrm{diag}(\mu_u, \mu_d) = \delta_{\mu,0}(\tfrac{1}{3}\mu_B + \tfrac{1}{2}\mu_I, \tfrac{1}{3}\mu_B - \tfrac{1}{2}\mu_I) \,, \\
f &\sim f_\pi \qquad\qquad m \sim m_\pi
\end{aligned}
$$

- **Static Hamiltonian**

$$
\mathcal{H}^{\mathrm{static}} = \frac{1}{8} f^2 \mu_I^2\,\mathrm{Tr}\left[\tau_3 \Sigma \tau_3 \Sigma^\dagger - 1\right] - \frac{1}{2} f^2 m^2\,\mathrm{Tr}\left[\Sigma + \Sigma^\dagger\right]
$$

— **Rotated ground state and pion condensation** [3]

$$\Sigma = \Sigma_0 \cos\alpha + \hat{\phi}_i \tau_i \sin\alpha \, ,$$
$$\mathcal{H}^{\rm static} = -f^2 m^2 \cos\alpha + \frac{1}{2}\mu_I^2 m^2 \sin^2\alpha (\hat{\phi}_1^2 + \hat{\phi}_2^2) \, .$$

— **Leading-order Lagrangian**

$$\mathcal{L}_2^{\rm static} = f^2 m^2 \cos\alpha + \frac{1}{2} f^2 \mu_I^2 \sin^2\alpha \, ,$$
$$\mathcal{L}_2^{\rm linear} = f\left(-m^2 \sin\alpha + \mu_I^2 \cos\alpha \sin\alpha\right)\phi_1 + f\mu_I \sin\alpha \partial_0 \phi_2 \, ,$$
$$\mathcal{L}_2^{\rm quadratic} = \frac{1}{2}(\partial_\mu \phi_a)(\partial^\mu \phi_a) + \mu_I \cos\alpha(\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1)$$
$$-\frac{1}{2}\Big[(m^2 \cos\alpha - \mu_I^2 \cos 2\alpha)\phi_1^2 + (m^2 \cos\alpha - \mu_I^2 \cos^2\alpha)\phi_2^2$$
$$+(m^2 \cos\alpha + \mu_I^2 \sin^2\alpha)\phi_3^2\Big] \, .$$

— **Goldstone mode**

$$E_{\pi^+} = \sqrt{\frac{\mu_I^4 - m_\pi^4}{3m_\pi^4 + \mu_I^4}}\, p + \mathcal{O}(p^2) \, .$$

[3] Son and Stephanov Phys. Rev. Lett. 86 592 (2001)

— **Ground state** [4]

$$
\begin{aligned}
\cos\alpha &= 1\,, \mu_I < m\,, \quad \text{(vacuum phase -Silver Blaze property)}\\
\cos\alpha &= \frac{m^2}{\mu_I^2}\,, \mu_I > m\,, \quad \text{(pion -condensed phase)}\\
\Sigma_\alpha &= A_\alpha \Sigma_0 A_\alpha\,,\\
A_\alpha &= e^{i\frac{\alpha}{2}(\hat{\phi}_1\tau_1+\hat{\phi}_2\tau_2)} = \cos\frac{\alpha}{2} + i(\hat{\phi}_1\tau_1+\hat{\phi}_2\tau_2)\sin\frac{\alpha}{2}\,,
\end{aligned}
$$

— **Fluctuations and rotated broken generators** [5]

$$
\begin{aligned}
\Sigma &= L_\alpha \Sigma_\alpha R_\alpha^\dagger\,,\\
L_\alpha &= A_\alpha U A_\alpha^\dagger\,,\\
R_\alpha &= A_\alpha^\dagger U^\dagger A_\alpha\,,\\
U &= e^{i\frac{\phi_a\tau_a}{2f}}\,.
\end{aligned}
$$

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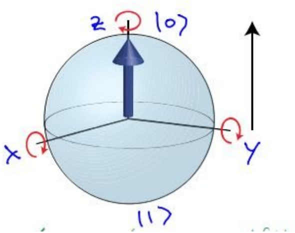
[4] D. Son and M. A. Stephanov, Phys.Rev. Lett. 86, 592 (2001)

[5] K. Splittorff D.T. Son, M. A. Stephanov, Phys.Rev. D64 (2001) 016003

— **Visualizing it via $SO(3) \to SO(2)$**



— **Broken generators**

$$U \quad = \quad e^{i\pi_1 J_1 + i\pi_2 J_y} \ .$$

— **Rotation around the x-axis**

$$R_x \left[i\pi_1 J_x + i\pi_2 J_y \right] R_x^{-1} \quad = \quad i\pi_1 J_x + i\pi_2 J_y \cos\theta - i\pi_2 J_z \sin\theta \ .$$

— **Naive parametrization**

$$\Sigma_{\text{wrong}} = U \Sigma_\alpha U = U A_\alpha \Sigma_0 A_\alpha U \ .$$

— **Use of unrotated generators yields noncanoical kinetic term (not problematic) and divergences that cannot be cancelled by standard counterterms (disaster)**

— **Next-to-leading order Lagrangian**

$$\mathcal{L}_4 = \frac{1}{4}l_1\left(\mathrm{Tr}\left[D_\mu\Sigma^\dagger D^\mu\Sigma\right]\right)^2 + \frac{1}{4}l_2\,\mathrm{Tr}\left[D_\mu\Sigma^\dagger D_\nu\Sigma\right]\mathrm{Tr}\left[D^\mu\Sigma^\dagger D^\nu\Sigma\right] + \frac{1}{16}(l_3+l_4)m^4\left(\mathrm{Tr}\left[\Sigma+\Sigma^\dagger\right]\right)^2 + \frac{1}{8}l_4 m^2\mathrm{Tr}\left[D_\mu\Sigma^\dagger D^\mu\Sigma\right]\mathrm{Tr}[\Sigma+\Sigma^\dagger]$$

— **Renormalization of parameters**

$$l_i = l_i^r(\Lambda) - \frac{\gamma_i}{2(4\pi)^2}\left[\frac{1}{\epsilon}+1+\ln\left(\frac{\Lambda^2}{m^2}\right)-\bar{l}_i\right] ,$$

— **Static part**

$$\mathcal{L}_4^{\mathrm{static}} = (l_1+l_2)\mu_I^4\sin^4\alpha + l_4 m^2\mu_I^2\cos\alpha\sin^2\alpha + (l_3+l_4)m^4\cos^2\alpha ,$$

— **Effective potential at NLO**

$$V_1 = V_{1,\pi^+} + V_{1,\pi^-} + V_{1,\pi^0} = \tfrac{1}{2}\int_p\left(E_{\pi^+}+E_{\pi^-}+E_{\pi^0}\right)$$

— **Isolate divergences by adding and subtracting divergent terms that can be calculated in dimensional regularization**

— **Renormalized effective potential**

$$
\begin{aligned}
V_{\rm eff} \;=\;& -f^2 m^2 \cos\alpha - \frac{1}{2} f^2 \mu_I^2 \sin^2\alpha \\
& -\frac{1}{4(4\pi)^2}\left[\frac{3}{2} - \bar{l}_3 + 4\bar{l}_4 + \log\left(\frac{m^2}{\tilde{m}_2^2}\right) + 2\log\left(\frac{m^2}{m_3^2}\right)\right] m^4 \cos^2\alpha \\
& -\frac{1}{(4\pi)^2}\left[\frac{1}{2} + \bar{l}_4 + \log\left(\frac{m^2}{m_3^2}\right)\right] m^2 \mu_I^2 \cos\alpha \sin^2\alpha \\
& -\frac{1}{4(4\pi)^2}\left[1 + \frac{2}{3}\bar{l}_1 + \frac{4}{3}\bar{l}_2 + 2\log\left(\frac{m^2}{m_3^2}\right)\right] \mu_I^4 \sin^4\alpha \\
& + V^{\rm fin}_{1,\pi^+} + V^{\rm fin}_{1,\pi^-}\;.
\end{aligned}
$$

Results

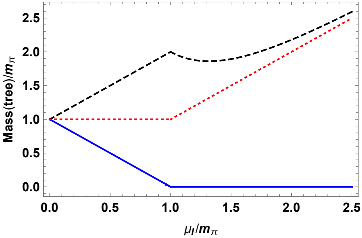
— **Low-energy constants**

$$\bar{l}_1 = -0.4 \pm 0.6 \,, \qquad \bar{l}_2 = 4.3 \pm 0.1 \,,$$
$$\bar{l}_3 = 2.9 \pm 2.4 \,, \qquad \bar{l}_4 = 4.4 \pm 0.2 \,.$$

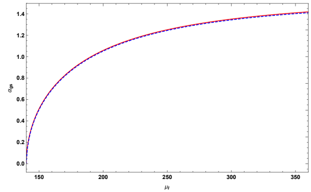
— **Parameters**

$$m_{\rm avg} = 136.87\,\text{MeV} \,, \qquad f_{\rm avg} = 79.70\,\text{MeV} \,,$$
$$m_{\rm min} = 135.31\,\text{MeV} \,, \qquad f_{\rm min} = 80.14\,\text{MeV} \,,$$
$$m_{\rm max} = 138.61\,\text{MeV} \,, \qquad f_{\rm max} = 79.23\,\text{MeV} \,.$$

— **Quasi-particle masses (Leading order)**



— **Minimizing with respect to α**



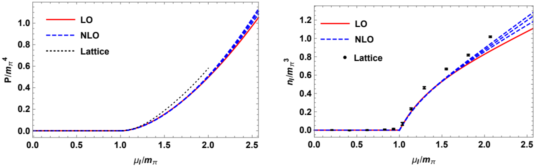
— **Expansion around $\alpha = 0$ to obtain Landau-Ginzburg free energy**

$$
\begin{aligned}
V_{\rm eff}^{\rm LG} &= V_{\rm eff}(\alpha=0) + \frac{1}{2}m^2 f^2 \left[1 - \frac{m^2}{2(4\pi)^2 f^2}\left(\bar{l}_3 - 4\bar{l}_4\right)\right]\alpha^2 \\
&\quad - \frac{1}{2} f^2 \mu_I^2 \left[1 + \frac{2m^2}{(4\pi)^2 f^2}\bar{l}_4\right]\alpha^2 + \mathcal{O}\left(\alpha^4\right) \\
&= V_{\rm eff}(\alpha=0) + \frac{1}{2} f_\pi^2 \left[m_\pi^2 - \mu_I^2\right]\alpha^2 + \mathcal{O}\left(\alpha^4\right) \ , \\
m_\pi^2 &= m^2 \left[1 - \frac{m^2}{2(4\pi)^2 f^2}\bar{l}_3\right] \ .
\end{aligned}
$$

— **Second order transition exactly at $\mu_I = m_\pi$ since $\alpha_4 > 0$**

— Pressure and isospin density

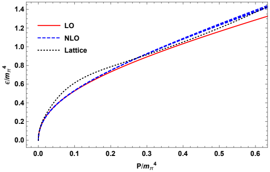
$$P = -V_{\rm eff}\,, \qquad n_I = -\frac{\partial V_{\rm eff}}{\partial \mu_I}$$



Lattice data from B. B. Brandt et al, Phys. Rev. D98 094510 (2018).

— Equation of state

$$\epsilon = n_I\mu_I - P\,.$$



Lattice data from B. B. Brandt et al, Phys. Rev. D98 094510 (2018).

Pion stars

— **Consists of a Bose condensate of charged pions**

— **Tolman-Oppenheimer-Volkov equations**

$$\frac{dM}{dr} = 4\pi r^2 \epsilon(r) \ ,$$
$$\frac{dP}{dr} = -G(\epsilon + P)\frac{M + 4\pi r^3 P}{r^2 - 2GrM} \ .$$

— **Impose charge neutrality**

$$n_Q = 0 \ .$$

— **Adding leptons**

$$\mathcal{L}_{\text{lepton}} = \bar{l}\left[i\not{\partial} + \mu_l \gamma^0 + m_l\right] l \ ,$$

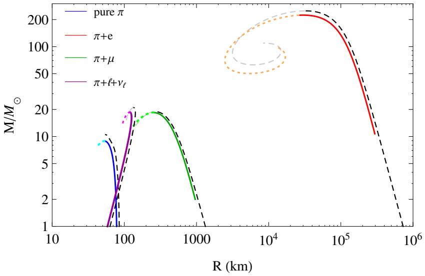
— Pion decay and chemical equilibrium

$$\begin{aligned}
\pi^+ &\leftrightarrow \mu^+ + \nu_\mu \ , \\
\pi^+ &\leftrightarrow e^+ + \nu_e \ , \\
\mu_I &= \mu_Q \ , \\
\mu_e &= -\mu_Q + \mu_{L_e} \ , \\
\mu_{\nu_e} &= \mu_{L_e} \ , \\
\mu_\mu &= -\mu_Q + \mu_{L_\mu} \ , \\
\mu_{\nu_\mu} &= \mu_{L_\mu} \ .
\end{aligned}$$

— More complicated TOV equation

$$\begin{aligned}
\frac{d\mu_e}{dr} &= -G\mu_e \frac{m + 4\pi r^3 P}{r^2 - 2Grm}\left[1 + \frac{\mu_I}{\mu_e} + \rho\left(\frac{\mu_\mu}{\mu_e} - 1\right)\right] \\
&\quad \times \left[1 + \frac{n'_e}{n'_I} + \frac{n'_\mu}{n'_I}\frac{d\mu_\mu}{d\mu_e} + \rho\left(\frac{d\mu_\mu}{d\mu_e} - 1\right)\right]^{-1} \ .
\end{aligned}$$

— Mass-radius relation



System	π	$\pi + \mu$	$\pi + e$	$\pi + l + \nu_l$
$M/M_\odot$	8.91	18.45	223.86	18.65
R [km]	53.06	252.08	2.724×10^4	118.9
P_c [MeV/fm^3]	20.6	0.23	4.56×10^{-7}	3.75
$\mu_{I,c}$ [MeV]	253.88	142.0928	140.0000851	140.905
P_s [MeV/fm^3]	0	0	0	0.43
n_ν [1/fm^3]	0	0	0	1.22×10^{-3}

Conclusions and outlook

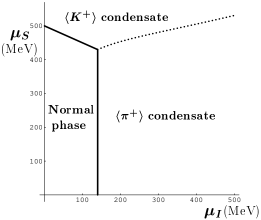
— **First calculation of thermodynamic functions at next-to-leading order in χpt in the pion-condensed phase**

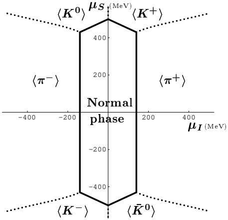
— **Good agreement with lattice data at $T = 0$. First precision test of χpt at NLO with nonzero μ_I**

— **Finite-temperature calculations straightforward. Order of transition?**

— **Three-flavor χpt NLO calculation on its way**

- Pion condensation and kaon condensation





XQCD 2020

- Trondheim, Norway July 22-24 2020.
- PhD school, Stavanger Norway, July 17-20 2020
- Confinement, Stavanger July 27-Aug 1 2020