Complex Langevin applied to chiral random matrix model in T-μ plane

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in collaboration with Y. Kikukawa

Plan

- Chiral random matrix as a model for QCD
- Complex Langevin eqn (CLE) results
  - at finite μ, T=0
  - at finite μ & T
- Summary

Sign problem in QCD at finite μ

- **Complex action from Fermion determinant:**

$$Z = \int DU\, D\bar{\psi}\, D\psi\; e^{-S} = \int DU\; e^{-S_B} \det D(U)$$

$$\left[\det D(U;\mu)\right]^* = \det D(U;-\mu^*) \quad \in \mathbb{C}$$

- **invalidates Importance Sampling; need for new methods!**

- Field complexification
  - Integration on *thimbles*
  - Complex Langevin equation

Cristoforetti-Di-Renzo-Scorzato (Aurora),
HF-Honda-Kato-Kikukawa-Komatsu-Sano
Tanizaki, Kanazawa-Tanizaki, Koike-Tanizaki,
HF-Kamata-Kikukawa,
Alexandru-Basar-Bedaque, ...,
Umeda-Fukuma, Mori-Kashiwa-Ohnishi, ...

Parisi, Klaudar, and many works in 80's

Aarts-Stamatescu,
Aarts-James-Seiler-Stamatescu,
Sexty,
Nagata-Nishimura-Shimasaki, ...

ChRM model at finite μ

A model with phase transition

Stephanov, PRL 76, 4472 (1996)

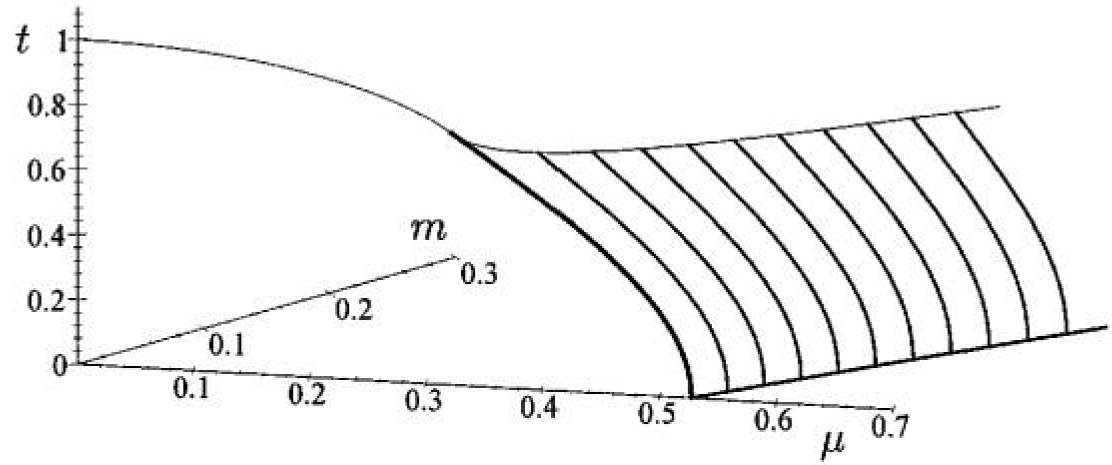
Halász et al. PRD 58:096007 (1998)

$$Z_N = \int [dW]\, e^{-N\Sigma^2 \operatorname{tr} W^+ W} \det D$$

$$D = \begin{pmatrix} m & iW + C \\ iW^+ + C & m \end{pmatrix} \qquad C = \begin{pmatrix} (\mu + it)\mathbf{1}_{N/2} & 0 \\ 0 & (\mu - it)\mathbf{1}_{N/2} \end{pmatrix}$$

W : $N \times N$ random matrix μ, t : deterministic parts

Phase diagram in large N (Nf=2)



ChRM and QCD

- ChRM becomes equivalent to QCD in the ε regime
- In CLE approach for fermion models,
  - Drift force K becomes singular at zeroes of det in complex z plane, which is problematic

$$z(t+\epsilon) = z(t) + \epsilon K(z) + \sqrt{\epsilon}\,\eta(t)$$

$$K(z) = -\frac{\partial S_b}{\partial z} + \frac{1}{\det D}\frac{\partial \det D}{\partial z}$$

Chiral Random Matrix Model

- We solve CLE for ChRM with Nf=1
- $\Sigma = 1$, $\varepsilon = 5\times 10^{-5}$, 10^6 samples; m=0.4, 0.1, 0.01, N=16 mainly

$$S = N\Sigma^2 \mathrm{tr}\, W^+ W - \log[\det D]$$

$$W = W_1 + i W_2; \quad W_{1,2} \in \mathbb{C}$$

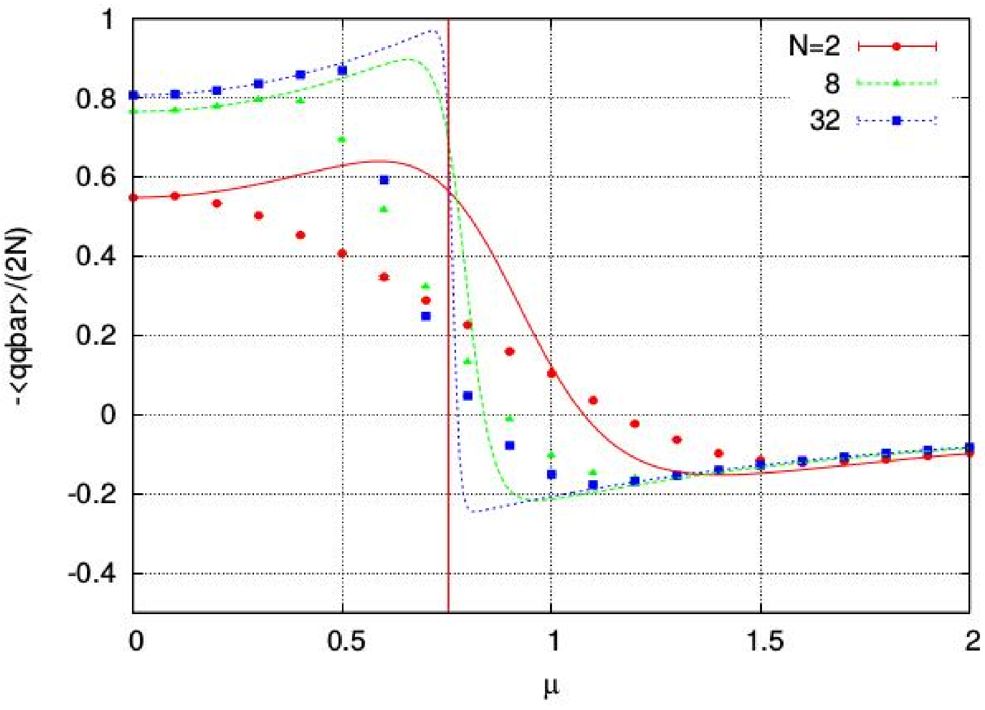
$$W_1(\theta+\epsilon) = W_1(\theta) + \epsilon K_1(\theta) + \sqrt{\epsilon}\eta_1(\theta)$$
$$W_2(\theta+\epsilon) = W_2(\theta) + \epsilon K_2(\theta) + \sqrt{\epsilon}\eta_2(\theta)$$

$$\langle O \rangle = \frac{1}{N_{\mathrm{sample}}} \sum_i O(W(\theta_i))$$

CLE result of ChRM at T =0

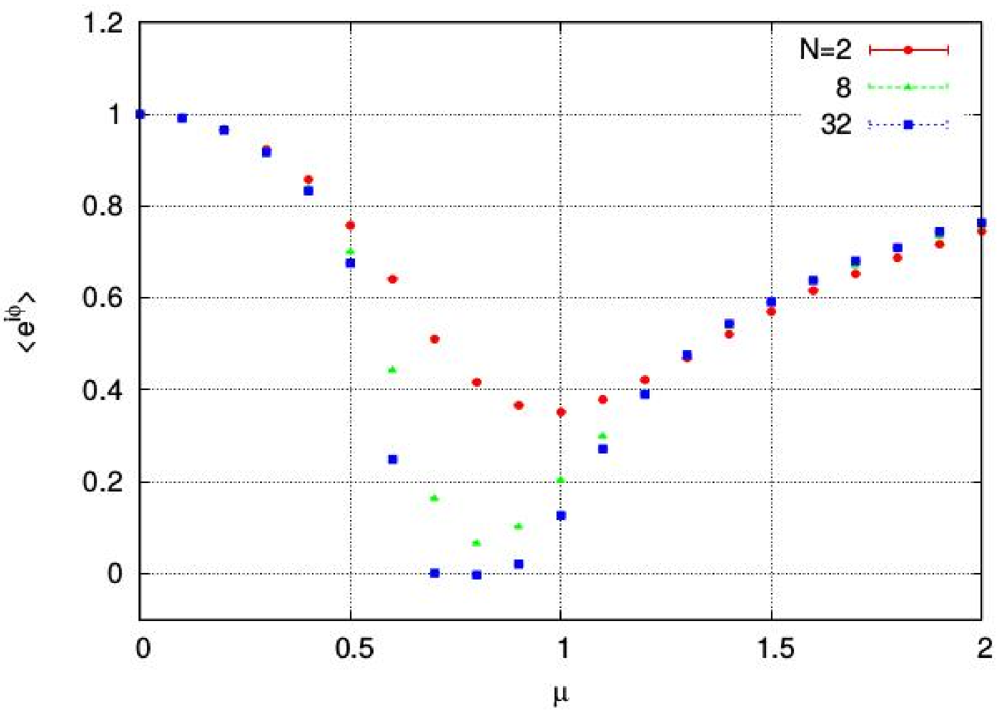
HF, Kikukawa, Sano,
HHIQCD2015,March, Kyoto

Quark condensate



Phase average

m=0.4

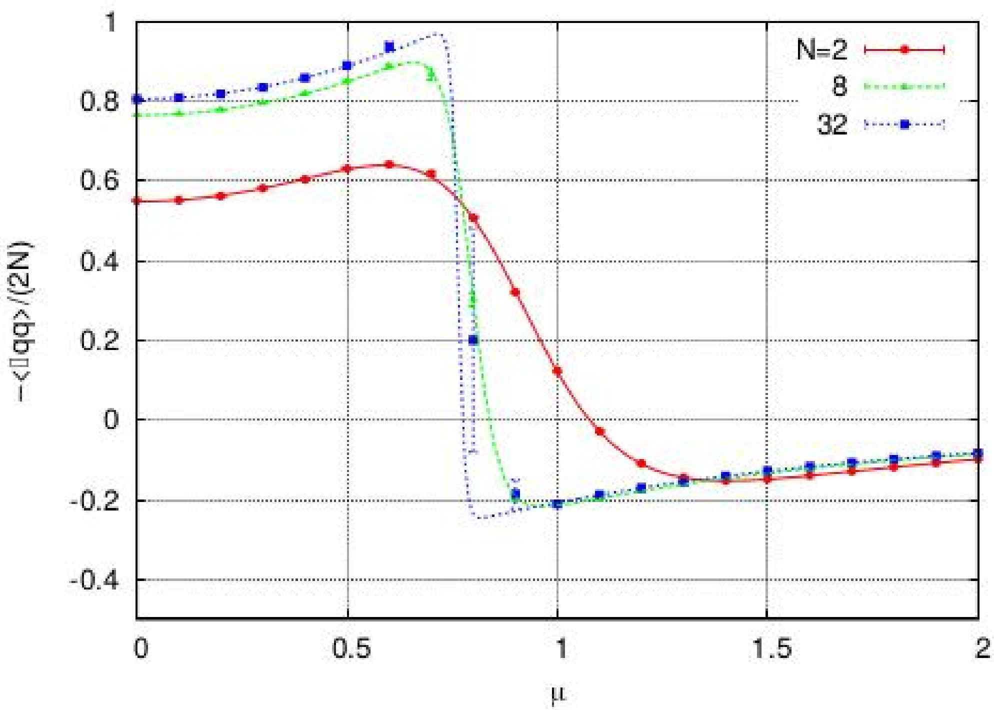


- Phase fluctuation is very mild for $N = 2$, but CLE already fails
- Failure is not directly related to severity of phase fluctuation

CLE result of ChRM at T =0

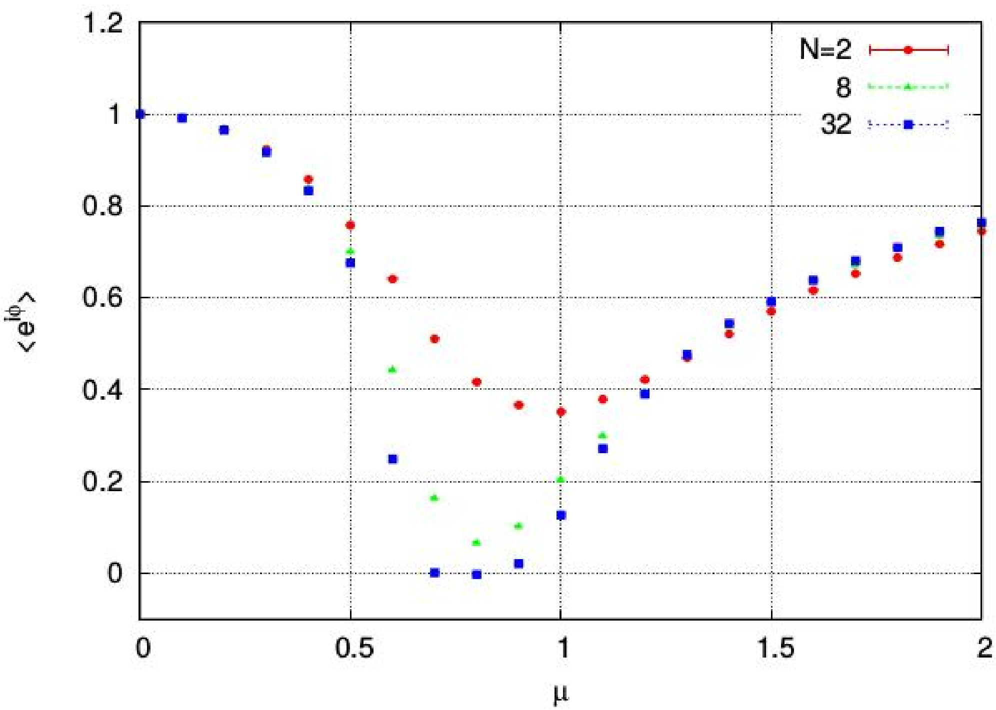
HF, Kikukawa, Sano,
HHIQCD2015,March, Kyoto

Quark condensate (re-weighting)



Phase average

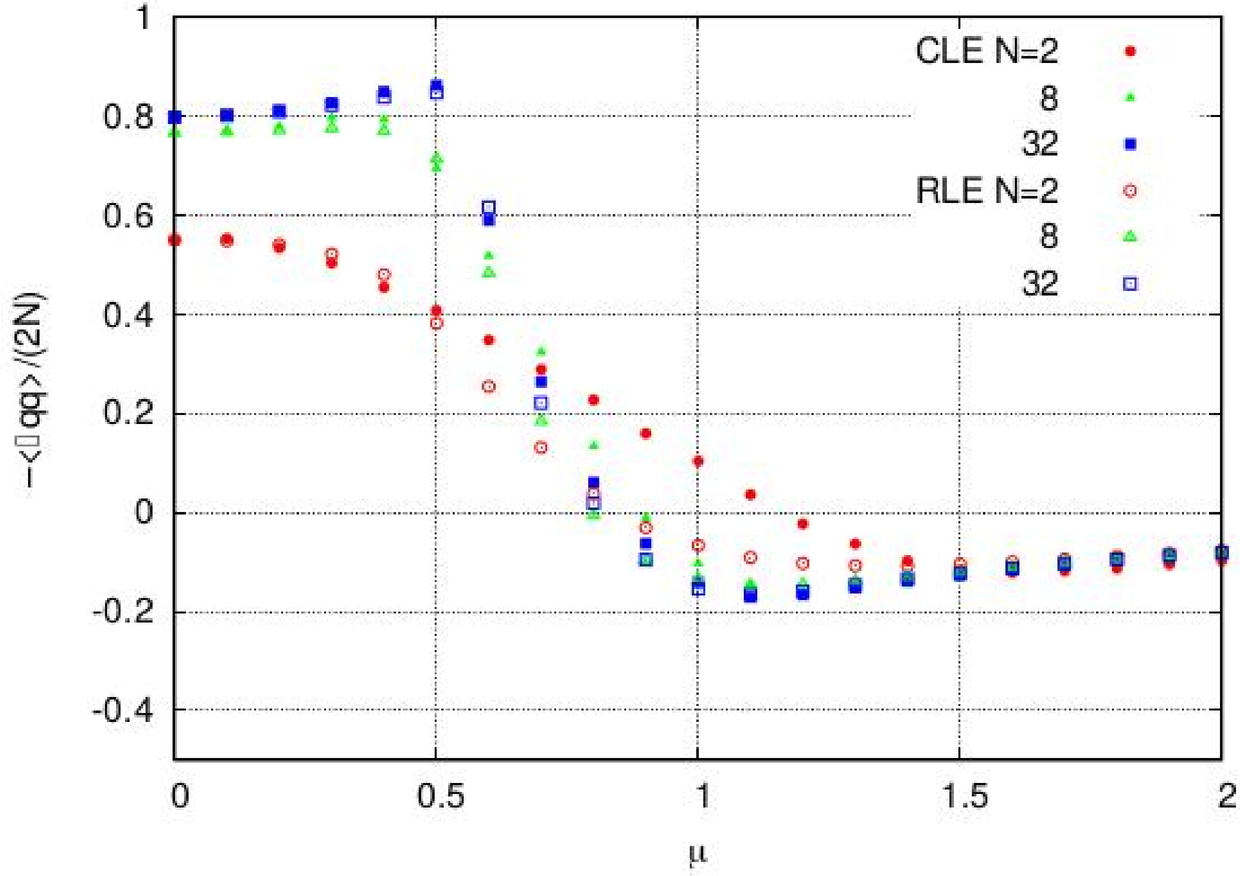
m=0.4



- Phase fluctuation is very mild for $N = 2$, but CLE already fails
- Re-weighting with $|\mathrm{ChRM}|$ works except for trans. region.

Comparison to |ChRM|

Quark condensate

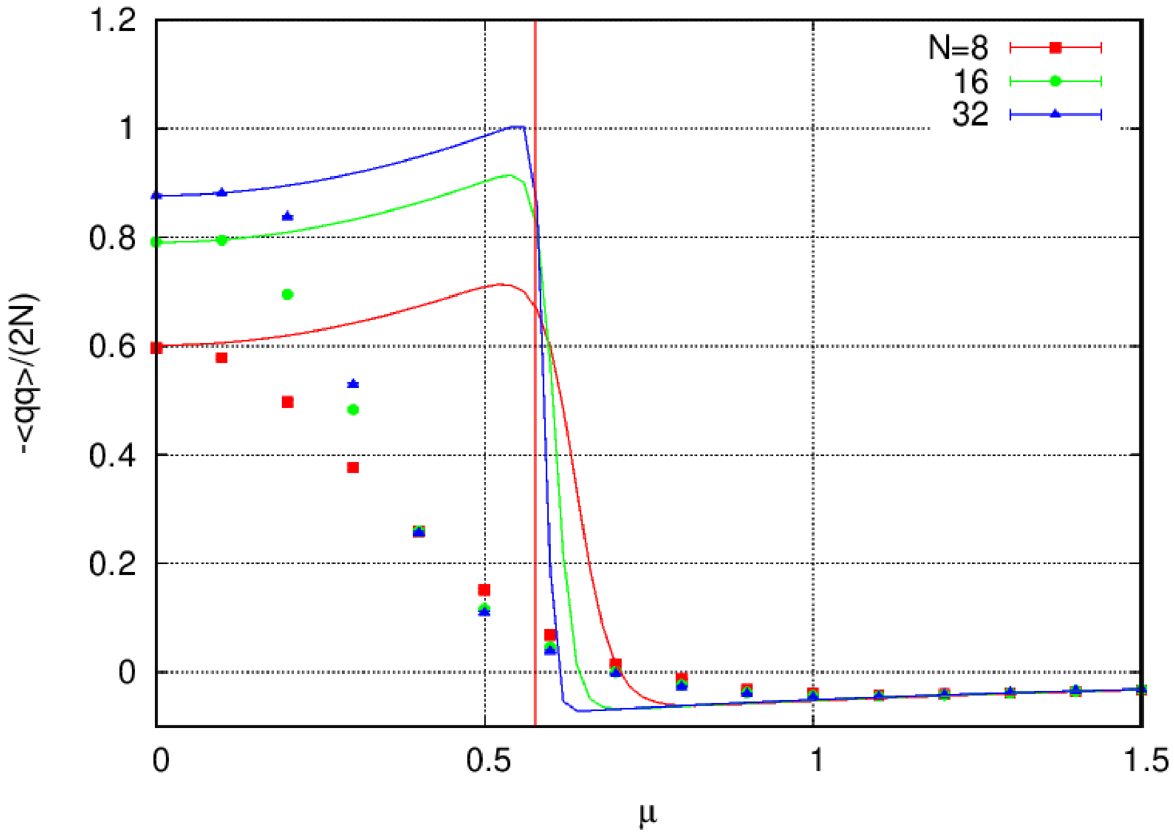


- For larger *N*, CLE results seem to converge to those of |ChRM|

J. Bloch et al. JHEP 1803 (2018) 015

Smaller quark mass m=0.1 at T =0

Quark condensate

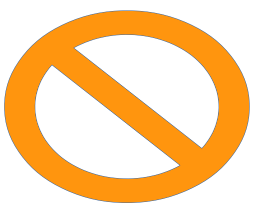


- Deviation from the exact solution starts at lower μ

Condition for correctness of CLE

Nagata-Nishimura-Shimasaki, PRD94(2016) no.11, 114515
refinement from Aarts-James-Seiler-Stamatescu

- CLE sampling in the region where drift becomes divergent, must be suppressed at least exponentially



- deep imaginary region
- vicinity of drift singularities

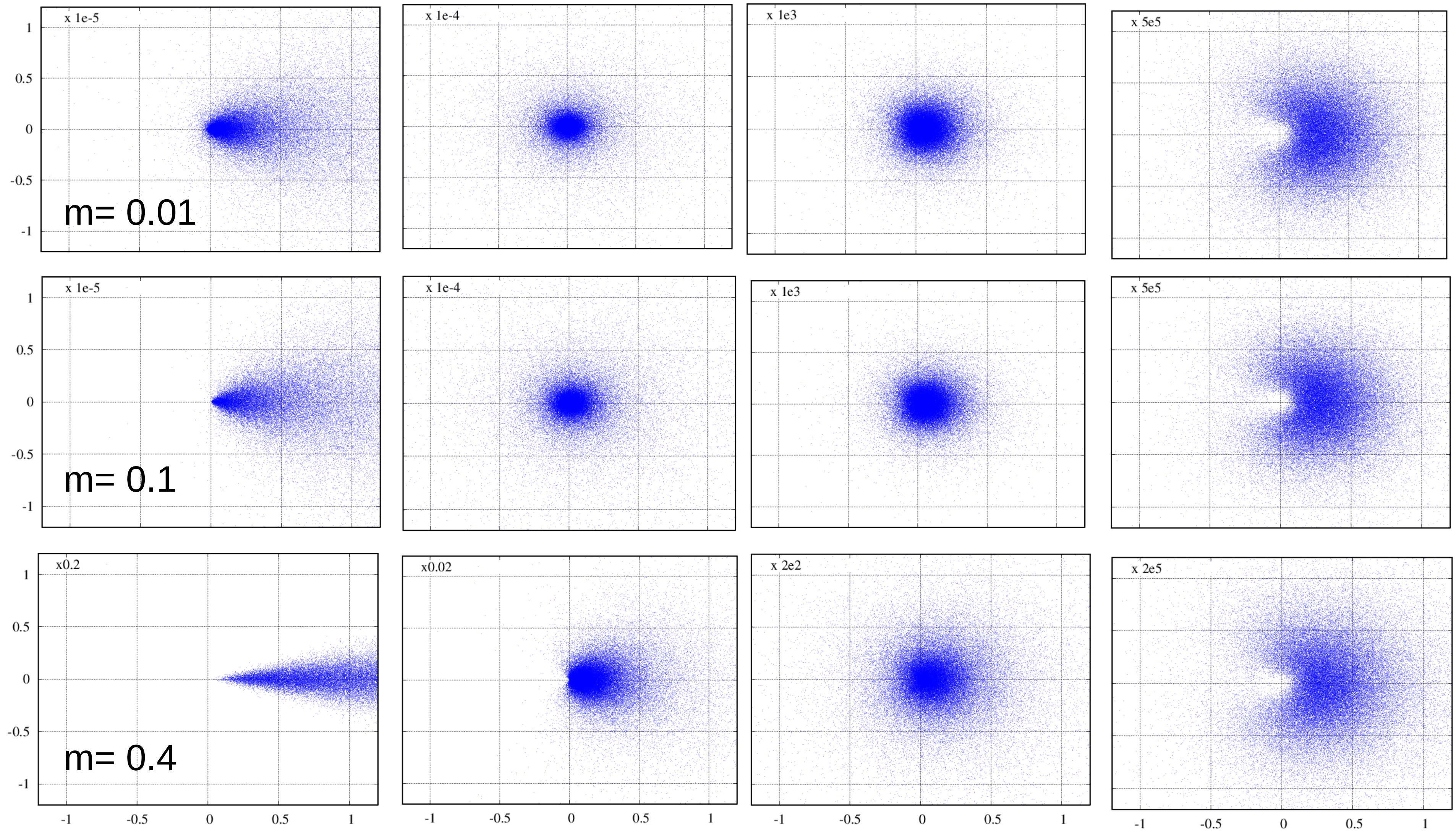
- If it's satisfied, then, for an integrable $O(z)$,

$$\boxed{\lim_{t\to\infty}\left[\lim_{\varepsilon\to 0}\int dx\,dy\,O(x+\mathrm{i}\,y)\,P(x,y;t)\right] = \frac{1}{Z}\int dx\,O(x)\,e^{-S}}$$

Det D(μ,T; W) of CLE samples

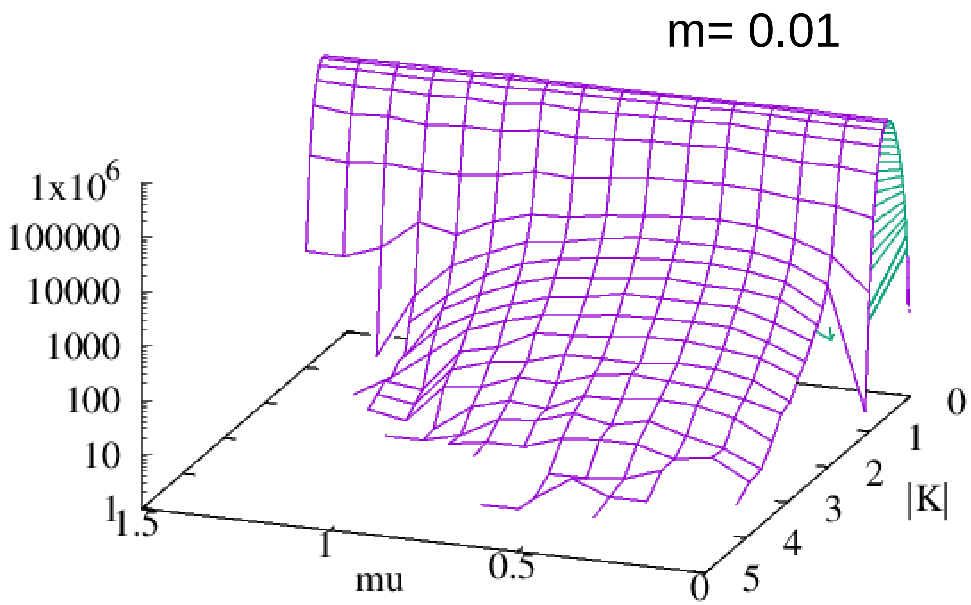
T=0

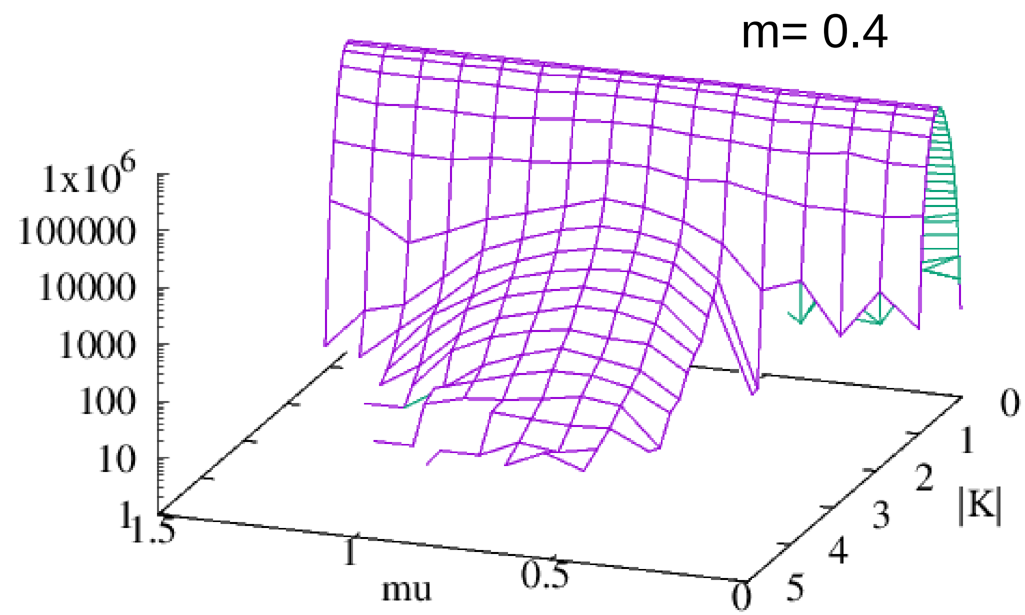
μ= 0.1 μ= 0.4 μ= 0.8 μ= 1.5



Histogram of |drift force| (T=0)

- Power-low tails are seen when CLE failes



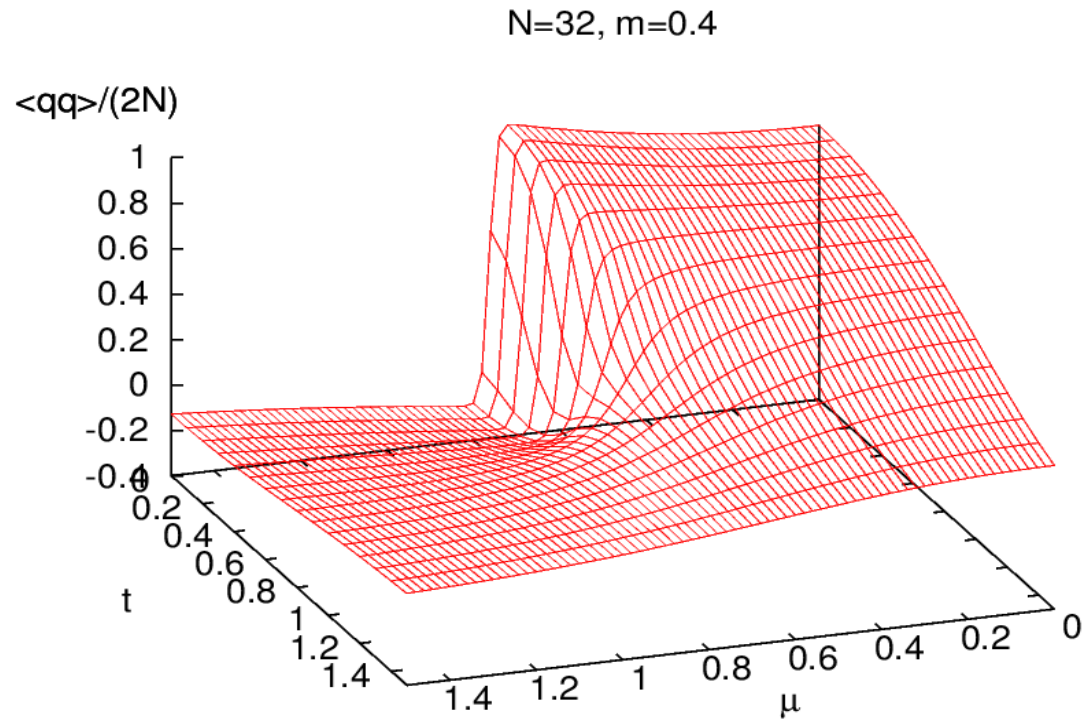


Extension to T ≠ 0

- **Analytic solution w/ finite *N* is available**

$$Z_{N,N_f=1} = \left(\frac{\pi}{\beta}\right)^{N^2} \frac{1}{\beta^N} \sum_{j_+,j_-=0}^{N/2} \binom{N/2}{j_+} \binom{N/2}{j_-} (\beta c_+^2)^{j_+} (\beta c_-^2)^{j_-} \; \ell! \, L(\ell, -\beta m^2)$$

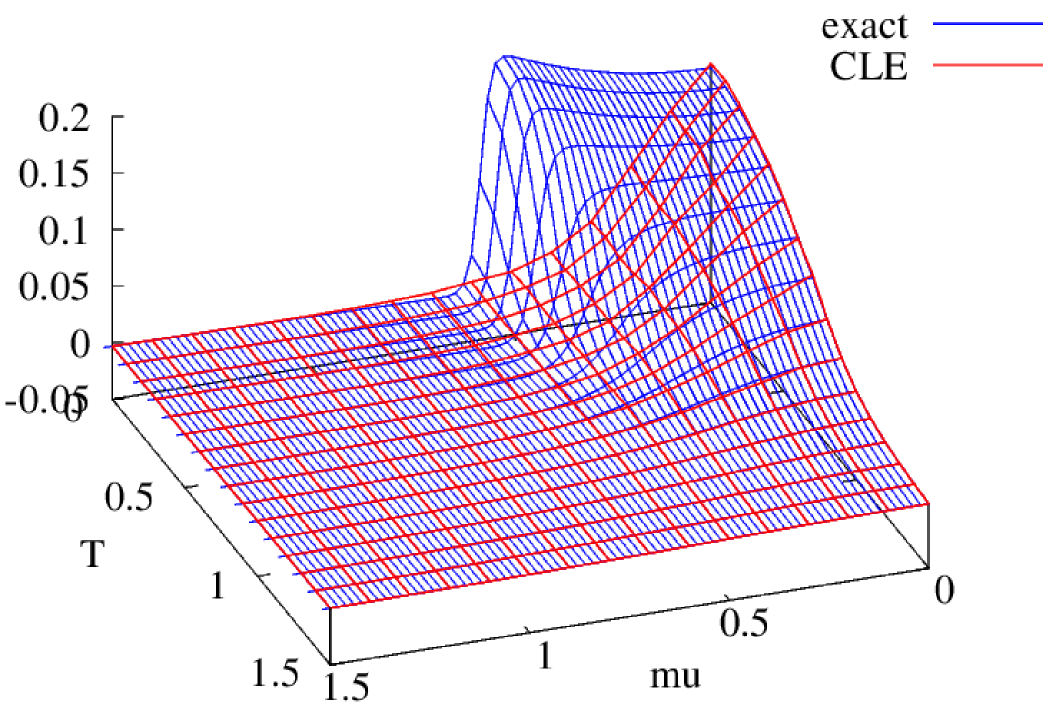
$c_\pm = t \pm \mathrm{i}\mu,\ \ell = N - j_+ - j_-$ L ... Laguerre Polynomial



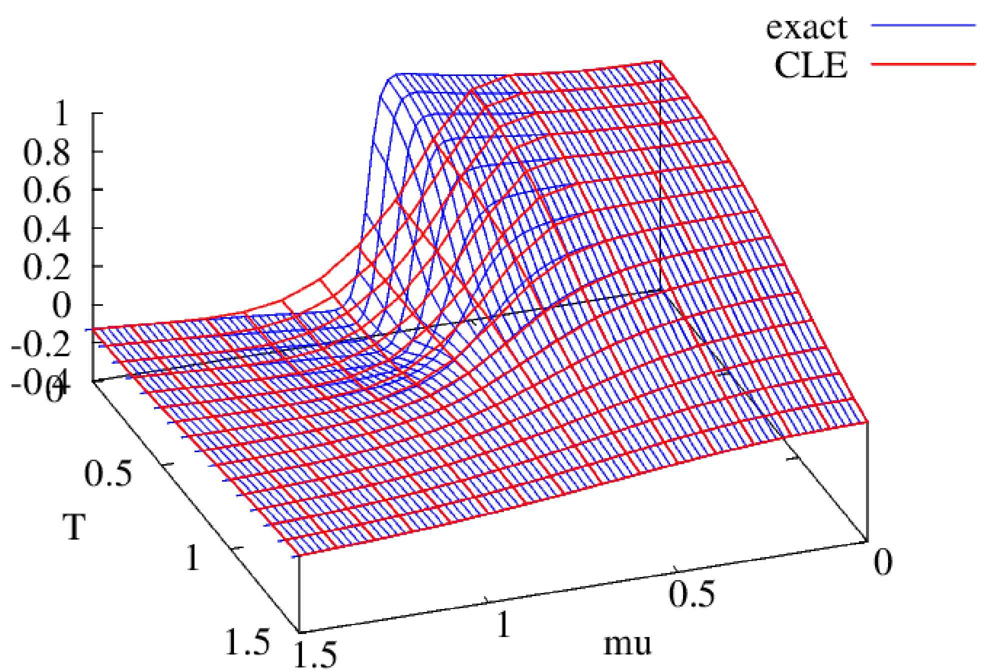
Phase diagram from CLE

N=16

m = 0.01

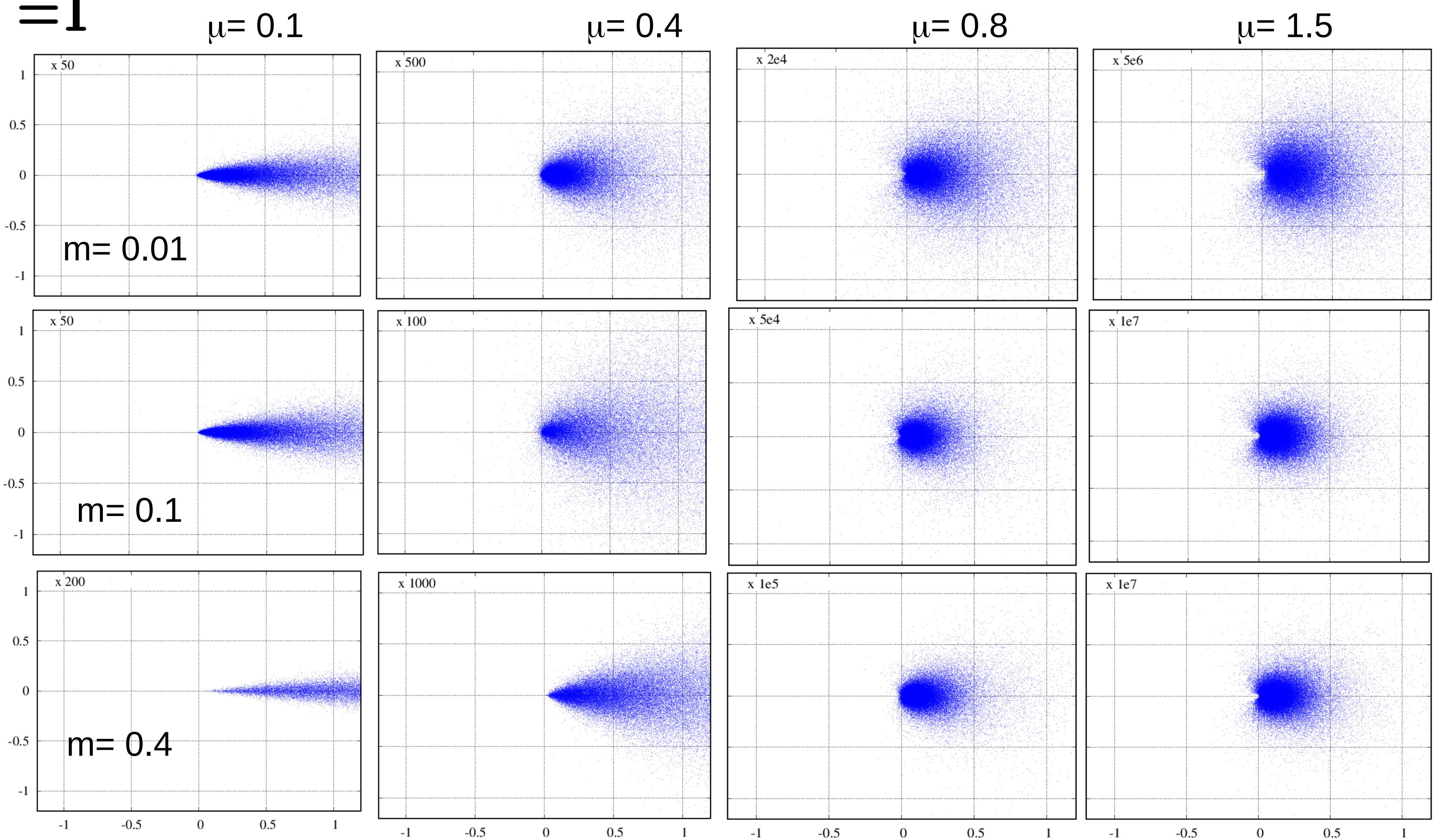


m = 0.4



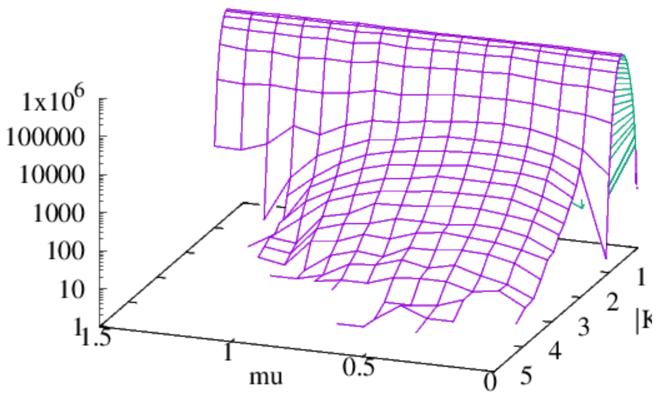
Det D(μ,T; W) of CLE samples

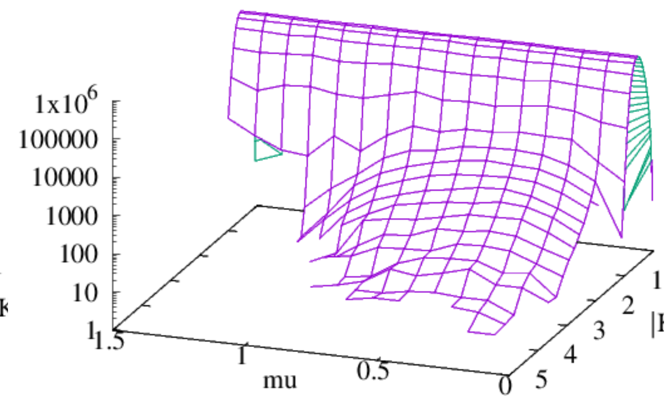
T=1

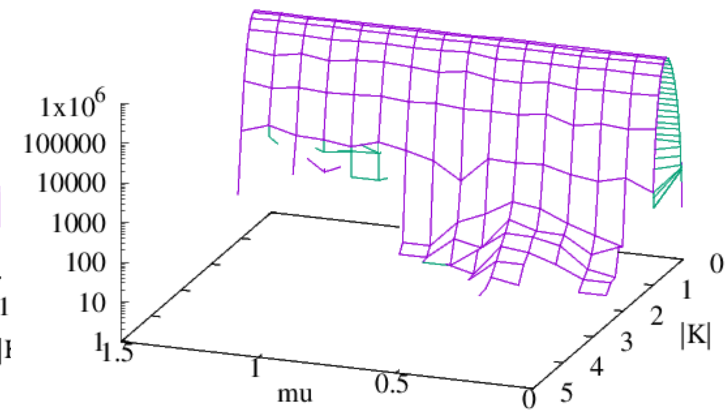


Histogram of |drift force|

m= 0.01

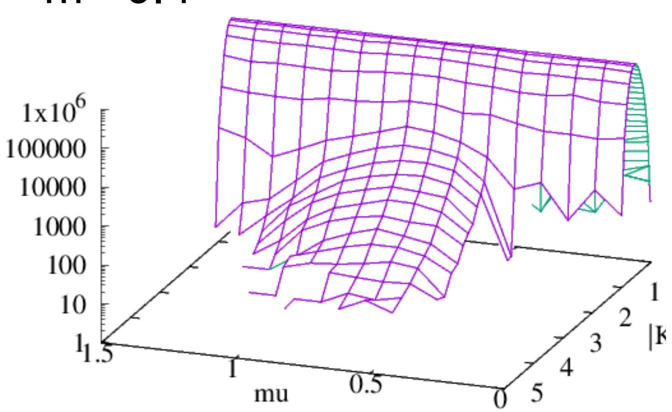


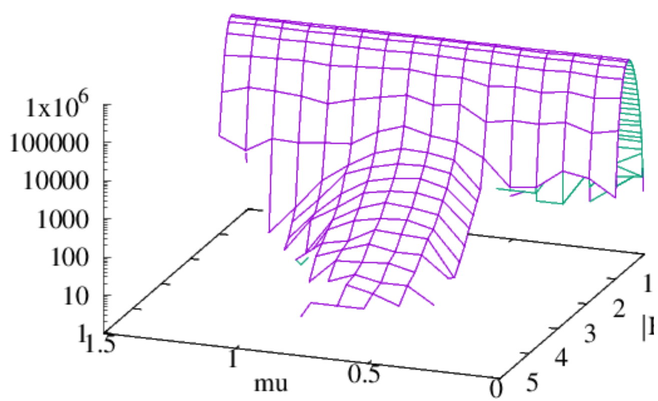


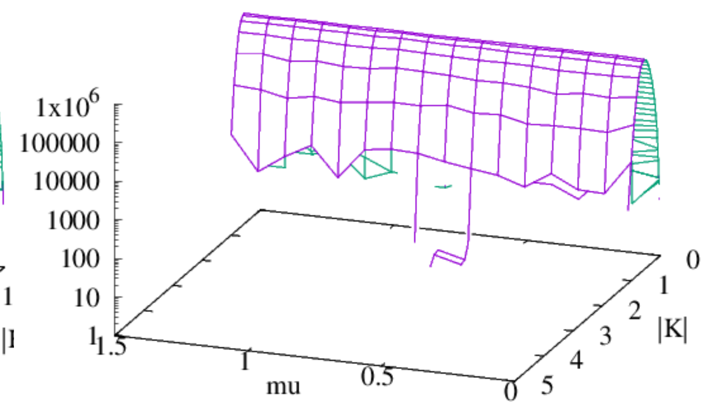


T=0 T=0.5 T=1

m= 0.4







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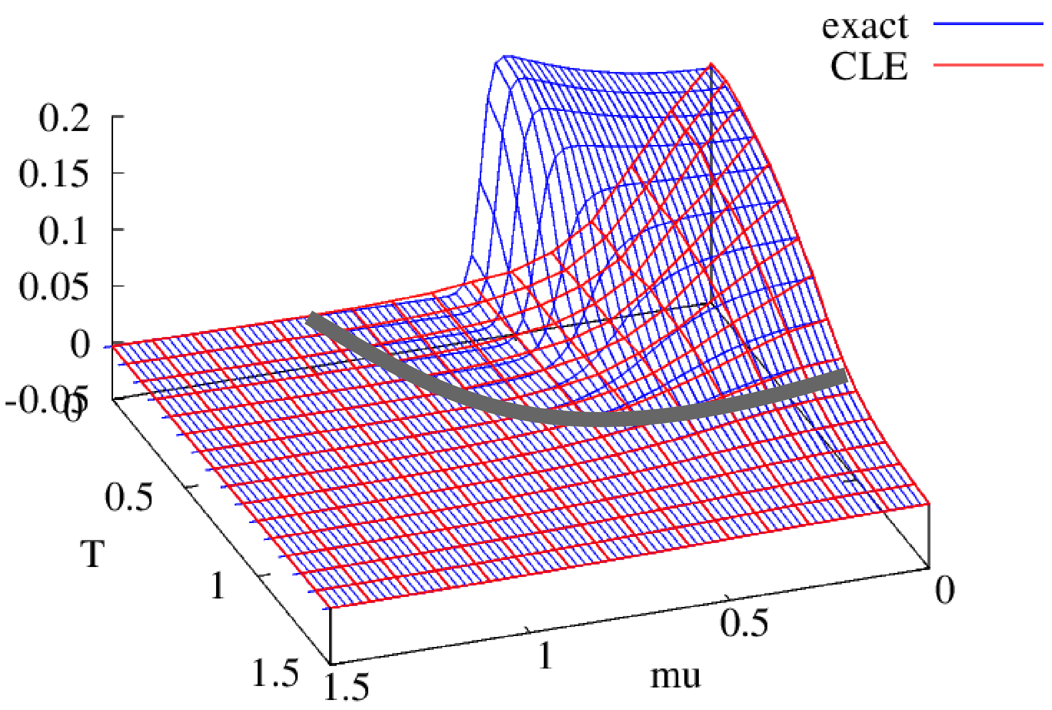
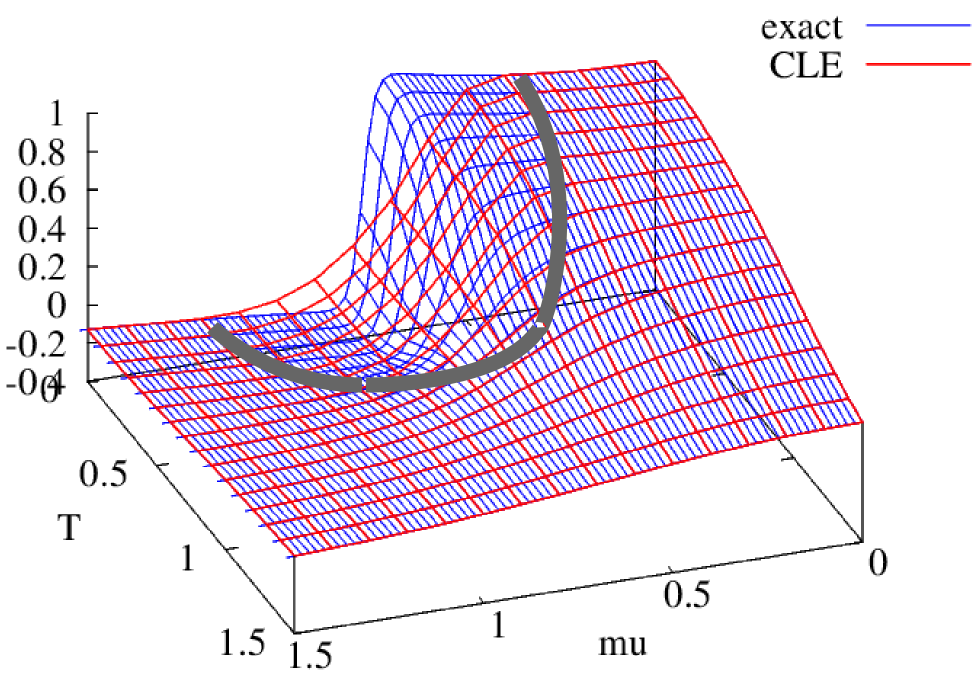


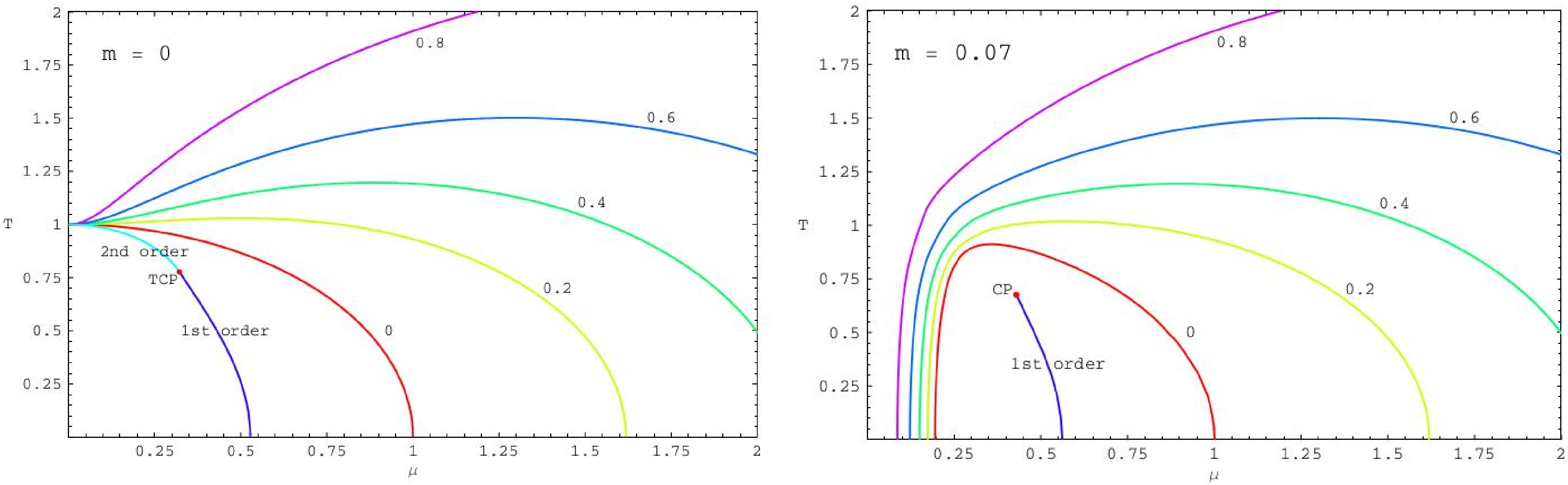
m = 0.4



Large N result of ChRM (Nf=2)

Han-Stephanov

- Contour map of $<\exp(2iq)>_{1+1^*}$



- π condensation for $\mu > m_\pi/2$ in |ChRM|, where μ enters in the support region of Dirac eigenvalues (w/ real *W*)

- CLE generates *complex W* configs, but the situation looks similar

Summary & outlook

- We have performed direct CLE simulations for ChRM model at finite μ and T
- Failure of CLE occurs consistently with correctness condition

  Nagata-Nishimura-Shimasaki, PRD94 (2016), 114515

  also Aarts-James-Seiler-Stamatescu

  – Cf. Various elaborate methods had been applied to this model

    J. Bloch et al. JHEP 1803 (2018) 015

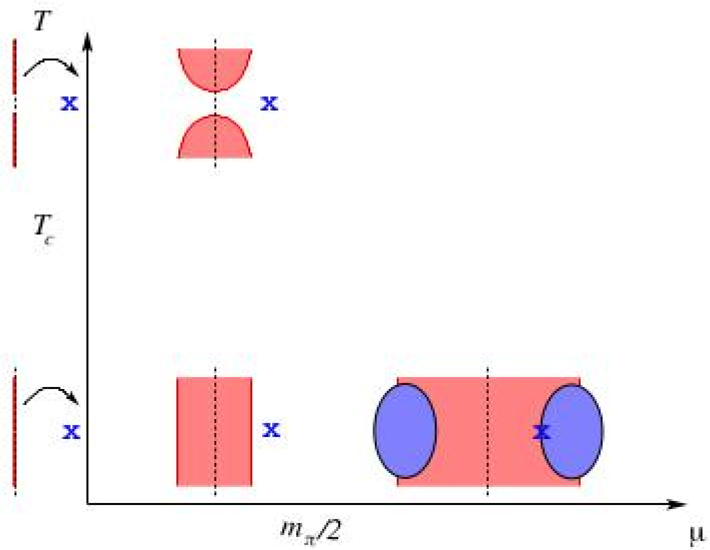
- Domain of failure in T-μ plane nearly overlaps with that of the π cond. region of phase quenched theory
- Outlook:

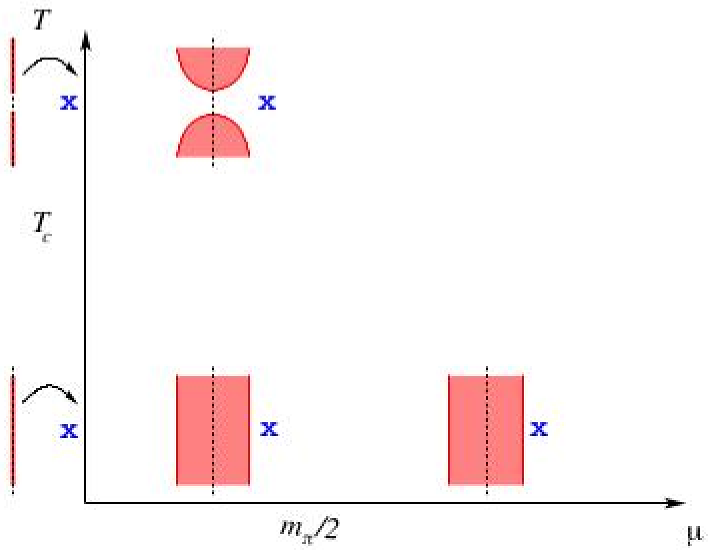
  J. Bloch et al. JHEP 1803 (2018) 015,
  K. Splittorff, 2015

  – Properties of the Dirac eigenvalues w/ complex gauge fields

Sketch of low-lying Dirac spectrum

K. Splittorff, 2014





Real gauge field:
eigenvalue density becomes
complex-valued in blue region

a senario for CLE to be successful

Chiral Random Matrix Model

- CLE

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HF, Kikukawa, Sano,
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$$W = W_1 + i W_2; \quad W_{1,2} \in \mathbb{C}$$

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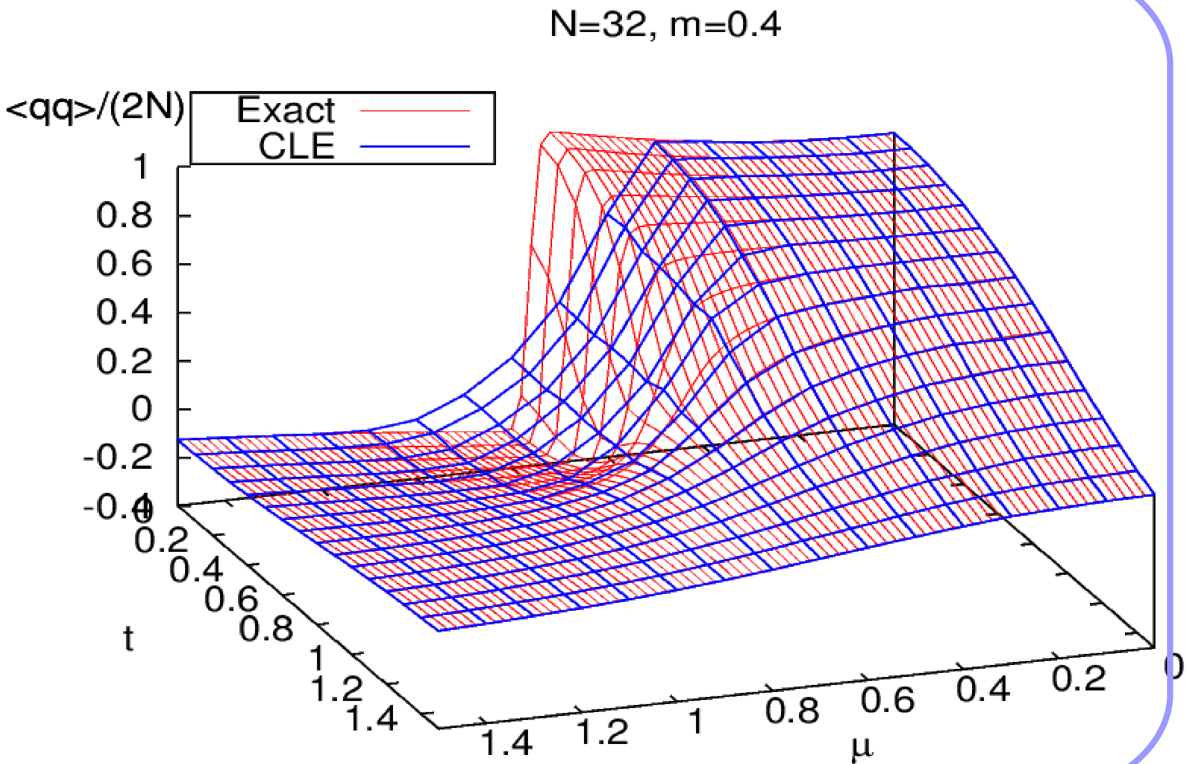
- fails in transition region
- works outside of it



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