On the multiple thimbles decomposition for the Thirring model

F. Di Renzo[1] and K. Zambello[1]

[1] *Dipartimento di Fisica dell'Università di Parma and INFN - Gruppo collegato di Parma*

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Thimble regularization

- Main idea: complexification of d.o.f. + deformation of integration domain

$$\int_{\mathcal{R}} dz^n\, O(z)\, e^{-S(z)} = \sum_\sigma n_\sigma e^{-iS^I_\sigma} \int_{\mathcal{J}_\sigma} dz^n\, O(z)\, e^{-S^R_\sigma}$$

- The thimble $\mathcal{J}_\sigma$ attached to a critical point p_σ is the union of the steepest ascent paths leaving p_σ

$$\frac{dz_i}{dt} = \frac{\partial \bar{S}}{\partial \bar{z}_i}, \text{ with i.c. } z_i(-\infty) = z_{\sigma,i}$$

Along the flow, the imaginary part of the action is constant.

- The tangent space at p_σ is spanned by the Takagi vectors, which can be found by diagonalizing the Hessian at the critical point

$$H(p_\sigma)v^{(i)} = \lambda_i^\sigma \bar{v}^{(i)}$$

Sampling thimbles

- A natural parametrization for a point on the thimble is $z \in \mathcal{J}_\sigma \leftrightarrow (\hat{n}, t)$, where $\hat{n}$ defines the direction on the tangent plane along which the path leaves the critical point and t is the integration time.

- In this parametrization, the thimbles decomposition of an expectation value $\langle O \rangle$ takes the form

$$\langle O \rangle = \frac{\sum_\sigma n_\sigma \int D\hat{n}\, 2\sum_i \lambda_i^\sigma n_i^2 \int dt\, e^{-S_{eff}(\hat{n},t)}\, O\, e^{i\omega(\hat{n},t)}}{\sum_\sigma n_\sigma \int D\hat{n}\, 2\sum_i \lambda_i^\sigma n_i^2 \int dt\, e^{-S_{eff}(\hat{n},t)}\, e^{i\omega(\hat{n},t)}}$$

where the effective action $S_{eff}(\hat{n},t) \equiv S_R(\hat{n},t) - ln|det\, V(\hat{n},t)|$ and the residual phase $e^{i\omega(\hat{n},t)} \equiv e^{i\, arg(det\, V(\hat{n},t))}$ are obtained from the parallel transported basis of the tangent space $V(\hat{n},t)$.

- When only one thimble contributes, one can rewrite $\langle O \rangle = \frac{\langle O e^{i\omega}\rangle_\sigma}{\langle e^{i\omega}\rangle_\sigma}$, having defined $\langle f \rangle_\sigma = \int D\hat{n}\, \frac{Z_{\hat{n}}}{Z_\sigma} f_{\hat{n}}$ with

$$Z_{\hat{n}} \equiv (2\sum_i \lambda_i^\sigma n_i^2) \int dt\, e^{-S_{eff}(\hat{n},t)}$$

$$Z_\sigma \equiv \int D\hat{n}\, Z_{\hat{n}}$$

$$f_{\hat{n}} \equiv \frac{1}{Z_{\hat{n}}}(2\sum_i \lambda_i^\sigma n_i^2) \int dt\, f(\hat{n},t) e^{-S_{eff}(\hat{n},t)}$$

$\rightarrow$ importance sampling, $P_{acc}(\hat{n}' \leftarrow \hat{n}) = min\left(1, \frac{Z_{\hat{n}'}}{Z_{\hat{n}}}\right)$.

- This can be generalized to more than one thimble:

$$\langle O \rangle = \frac{\sum_\sigma n_\sigma Z_\sigma \langle O e^{i\omega}\rangle_\sigma}{\sum_\sigma n_\sigma Z_\sigma \langle e^{i\omega}\rangle_\sigma}$$

Weights

How to compute the weights of the thimbles?

- If only two thimbles are relevant, one may give up prediction power on one observable and compute the relative weight from

$$\langle O \rangle = \frac{n_0 Z_0 \langle O e^{i\omega}\rangle_0 + n_{12} Z_{12} \langle O e^{i\omega}\rangle_{12}}{n_0 Z_0 \langle e^{i\omega}\rangle_0 + n_{12} Z_{12} \langle e^{i\omega}\rangle_{12}} = \frac{\langle O e^{i\omega}\rangle_0 + \alpha \langle O e^{i\omega}\rangle_{12}}{\langle e^{i\omega}\rangle_0 + \alpha \langle e^{i\omega}\rangle_{12}}$$

Applied to QCD-(0+1) in Ref. [1].

- One may also compute the semiclassical weights and then their corrections
  1. $\frac{Z_\sigma^G}{\sum_{\sigma'} Z_{\sigma'}^G}$ (semiclassical weights)
  2. $Z_\sigma^G = \int D\hat{n} \frac{Z_{\hat{n}}^G}{Z_{\hat{n}}} Z_{\hat{n}} = Z_\sigma \int D\hat{n} \frac{Z_{\hat{n}}^G}{Z_{\hat{n}}} \frac{Z_{\hat{n}}}{Z_\sigma} = Z_\sigma \langle \frac{Z_{\hat{n}}^G}{Z_{\hat{n}}} \rangle \rightarrow \frac{Z_\sigma}{Z_\sigma^G} = \langle \frac{Z_{\hat{n}}^G}{Z_{\hat{n}}} \rangle^{-1}$

From (1) and (2) one obtains $\frac{Z_\sigma}{\sum_{\sigma'} Z_{\sigma'}^G}$, which is what we want up to a normalization factor. Applied to HDQCD in Ref. [2].

(0 + 1) - dimensional Thirring model

- Let's consider the (0 + 1) - dimensional Thirring model

$$S = \beta \sum (1 - cos(z_n)) + log\, det D$$

$$det D = \frac{1}{2^{L-1}}(cosh(L\hat{\mu} + i\sum z_n) + cosh(L\hat{m}))\ ,\ \hat{\mu} \equiv a\mu\ ,\ \hat{m} = asinh(am)$$

It has been shown before that one thimble is not enough to capture the full content of the theory (see Ref. [3]; see also Ref. [4]).

- Can we collect contributions coming from more than one thimble with our approach?

[1] F. Di Renzo and G. Eruzzi, Phys.Rev. D97 (2018) no.1, 01450
[2] F. Di Renzo and K. Zambello, PoS LATTICE2018 (2018) 148
[3] H. Fujii, S. Kamata and Y. Kikukawa, JHEP 1511 (2015) 078
[4] A. Alexandru, G. Başar, P. F. Bedaque et al., JHEP 1605 (2016) 053

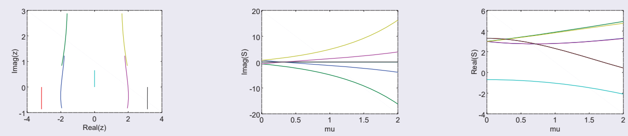
Critical points

- The critical points are found by imposing

$$\frac{\partial S}{\partial z_n} = \beta sin(z_n) - i\frac{sinh(L\mu + i\sum z_n)}{cosh(L\mu + i\sum z_n) + cosh(Lm)} = 0$$

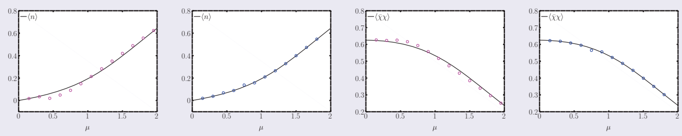
The second term depends on the fields only through the sum $s \equiv \sum z_n$, then it must be $sin(z_n) = sin(z)\ \forall n$ and z_n can be either z or $\pi - z$. Following Ref. [3] we define n_- as the number of flipped components.

- To find out which thimbles give relevant contributions we look at the S_I vs μ plot for possible Stokes phenomena, after which the intersection number can change. We focus on the critical points in the $n_- = 0$ sector for $L = 2$, $\beta = 1$ and $\mu = 0.0 \ldots 2.0$:

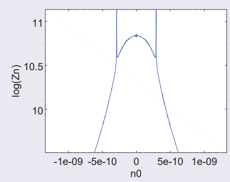


Numerical integration

- For $L = 2$ and $\beta = 1$ we have only 2 degrees of freedom and we can integrate numerically. Results from 1 (red data) and 3 thimbles (blue data):

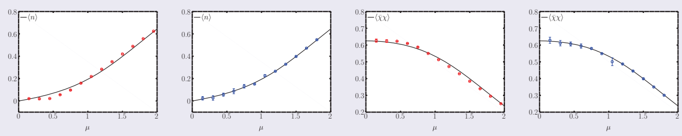


One can notice a strong dependence of Z_n on n_0. n_0 is the component of the initial displacement on the tangent space at the critical point associated to the larger Takagi value. Sharp peaks show up at given values of n_0. These observations appear to hold also for higher L.

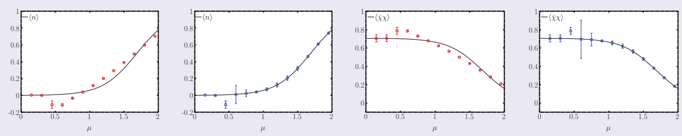


Preliminary results from MC

- Results for $L = 2$ and $\beta = 1$ from 1 (red data) and 3 thimbles (blue data):



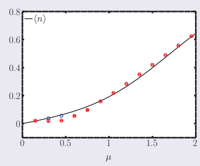
- Results for $L = 4$ and $\beta = 1$ from 1 (red data) and 3 thimbles (blue data):



Exploring alternative approaches

- The idea is to apply Taylor expansion in a region where there is a simpler thimble decomposition. The rationale for it: Stokes phenomena encode discontinuities in the thimble decomposition, not in the observables.

For $L = 2$ only one thimble gives a relevant contribution at $\beta = 1$, $\mu = 0.15$. Here we show the results for $\beta = 1$ and $\mu = 0.30$, 0.45 computed from a Taylor expansion at $\mu = 0.15$ (blue data) on that single thimble.



Conclusions

- We have studied the $(0 + 1)$-dimensional Thirring model for $L = 2$ and $L = 4$ at a strong coupling $\beta = 1$.
- Discrepancies between the analytical solution and the results from one thimble simulations seem to disappear keeping into account the two sub-leading thimbles (actually only one, due to a simmetry).
- We started applying Taylor expansions (this is a one thimble computation!)